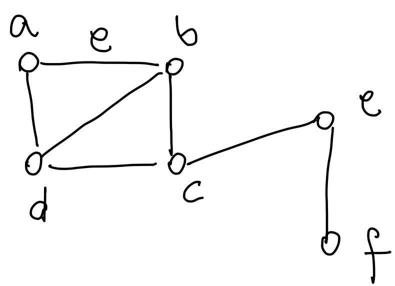
**Defn:** A graph is a pair of sets $(V, E)$ st $E \subseteq \{\{u,v\} : u, v \in V, u \neq v\}$

$V$: vertex set $\qquad E$: edge set.

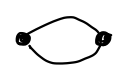
with $|V| < \infty$.

**Notation:** $G = (V(G), E(G)) = (V, E)$.

**Drawing a graph:**

$G =$ 


- $V(G) = \{a, b, c, d, e, f\}$
- $E(G) = \{ab, ad, bd, cd, bc, ce, ef\}$
- $ab = \{a, b\} = ba$

- NO LOOPS , NO MULTIPLE EDGES , NO DIRECTED EDGES 

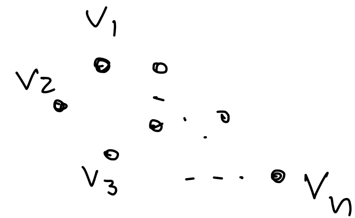
## Our dictionary

Let $G=(V,E)$ be a graph.

- For $e=xy \in E$, $x$ and $y$ are the endpoints of $e$.
- $x,y \in V(G)$ are adjacent if $xy \in E(G)$.
- $e \in E(G)$ is incident with $v \in V(G)$ if $x \in e$.
- $y$ is a neighbour of $x$ if $xy \in E(G)$. Notation: $x \sim y$.
- $N_G(x) = \{y : xy \in E(G)\}$ is the neighbourhood of $x$.
- $d_G(x) = |N_G(x)|$ is the degree of $x$.
- $v(G) := |V(G)|$ is the order and $e(G) := |E(G)|$ is the size.

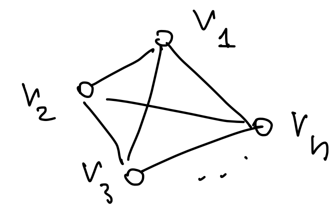
# Special types of graphs

- Empty graph on n vtxs :

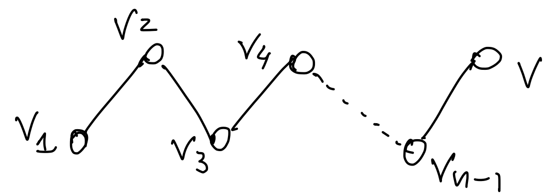



NO EDGE.

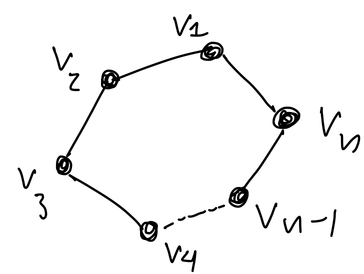
- Complete graph on n vtxs : $K_n =$




- Path of length n : $P_n =$




- Cycle of length n : $C_n =$



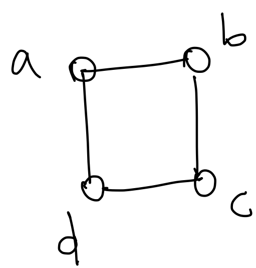
# Graph isomorphism.

**Defn:** Let $H$ and $G$ be graphs. $f : V(G) \to V(H)$ is an isomorphism if $uv \in E(G) \iff f(u)f(v) \in E(H)$ $\forall u, v \in V(G)$.

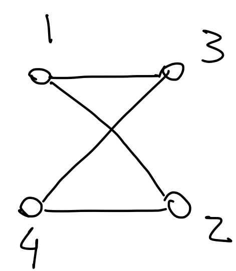
$H$ and $G$ are isomorphic if $\exists$ an isomorphism $f : V(G) \to V(H)$.

Notation : $H \cong G$.

When $H = G$, an isomorphism $f : V(G) \to V(H)$ is called automorphism




$\cong$




$(a, b, c, d) \mapsto (1, 2, 3, 4)$ is isomorphism.

Abuse of notation: "$P_n \subseteq G$" means $\exists H \subseteq G$ st $H \cong P_n$.

# Key definitions

Let $H$ and $G$ be graphs.

① $H$ is a subgraph of $G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Notation : $H \subseteq G$.

② For $U \subseteq V(G)$, $G[U] = (U, E)$ with $E = \{xy : xy \in E(G), x, y \in U\}$

$G[U]$ = the subgraph of $G$ induced by $U$.

③ $H \subseteq G$ is an induced subgraph if $H = G[U]$ for some $U \subseteq V(G)$.

④ Complement of $G$ : $\overline{G} = (V(G), E)$ with $E = \{xy : xy \notin E(G), x, y \in V(G)\}$

## Useful notations

① For $U \subseteq V(G)$, $G - U = G[V(G) \setminus U]$.

If $U = \{u\}$, we write $G - u$.

② For $F \subseteq E(G)$, $G - F = (V(G), E(G) \setminus F)$.

If $F = \{e\}$, we write $G - e$.

③ $\Delta(G) := \max d_G(u)$

④ $\delta(G) := \min d_G(u)$.

⑤ $d(G) = \frac{1}{n} \sum_{v \in V(G)} d_G(v)$.

The handshaking lemma: $\displaystyle\sum_{u \in V(G)} d(u) = 2e(G).$

proof:

Count $X = \#\{(x,e) : x \in e,\ x \in V(G),\ e \in E(G)\}$

$$\sum_{x \in V(G)} d(x) = X = 2e(G)$$

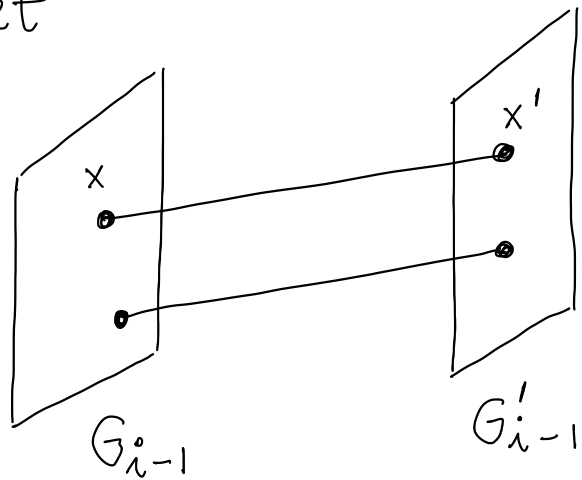
Every edge has 2 vtxs

x meets d(x) edges.

Lemma: $\forall\, G\ \exists\, H$ such that $H$ is $\Delta(G)$-regular and $G \subseteq H$.

proof: $G_0 = G$ and $G_0' \cong G$, $V(G_0) \cap V(G_0') = \phi$

For $i \geq 1$, set

$G_i =$ 


Connect $x$ to its twin $x'$ in $G'_{i-1}$ if $d_{G_{i-1}}(x) < \Delta(G)$.

STOP if $G_i$ is $\Delta(G)$-regular.