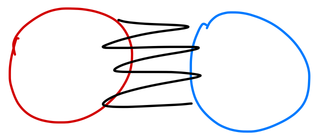
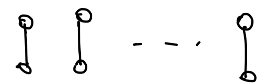
# Bipartite graphs and matchings.

**Defn**: $G$ is bipartite if $\exists\, A, B$ st $V(G) = A \cup B$ and

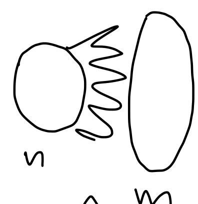
$$e(G[A]) = e(G[B]) = 0.$$

**Notation**: $G[A,B]$.

Alternatively: $\exists$  with no monochromatic edge!

**Examples of bipartite graphs**: even cycles, 

**Complete bipartite graph** $K_{n,m}$:




All edges between.

**Examples of non-bipartite graphs**: cliques, odd cycles.

<u>Lemma</u>: $G$ is bipartite $\iff C_{2\ell+1} \not\subseteq G \ \forall \ell \geq 1$.

<u>proof</u>: $(\Rightarrow)$ Odd cycles are not bipartite!

$(\Leftarrow)$ W.L.O.G assume $G$ is connected. Take $T \subseteq G$ spanning tree of $G$.

<u>Claim</u>: $T$ is bipartite.

<u>proof</u>: Take $r \in V(T)$ and let $c : V(T) \to \{0,1\}$ be given by

$$c(u) = (\mathrm{dist}_T(u,r) \pmod 2)$$

<u>Recall</u>: $uv \in E(T) \Rightarrow |\mathrm{dist}_T(u,r) - \mathrm{dist}_T(v,r)| = 1$

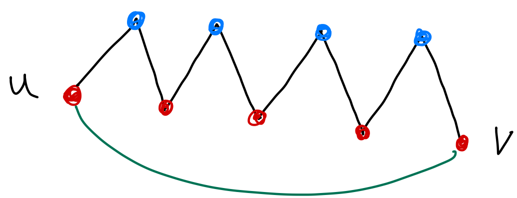
$\Rightarrow$ No edge is monochromatic. ——— $\blacksquare$

Let $(A,B)$ be the bipartition of $T$.

<u>Claim</u>: $(A,B)$ is also a bipartition for $G$.

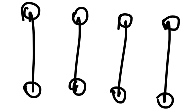
<u>proof</u>: Suppose by contradiction $\exists$ $uv \in E(G[A])$.

Let $P$ be the unique $u$-$v$ path in $T$.



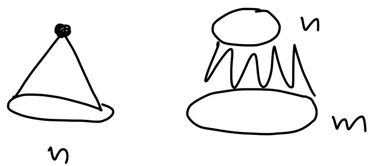

$\leftarrow$ # edges = even.

$\uparrow$ odd cycle! Contradiction $\blacksquare$

<u>Defn</u>: $M$ is a matching if $d_M(v) = 1 \ \forall\, v \in V(M)$. 

Size of $M = e(M)$.

Q/ What is the size of the largest matching we can find in a graph?




$\leftarrow$ largest matching $= n$.

<u>Defn</u>: $X \subseteq V(G)$ is a <u>vertex cover</u> if $e \cap X \neq \emptyset \ \forall\, e \in E(G)$.

<u>Obs</u>: $M$ max. matching and $X$ min. cover $\Rightarrow e \cap X \neq \emptyset \ \forall\, e \in E(M)$.

$\Rightarrow 2e(M) \geq |X| \geq e(M)$.

**Thm** (König) $G$ bipartite

$$\underbrace{\max\{e(M) : M \subseteq G \text{ matching}\}}_{\nu(G)} = \underbrace{\min\{|X| : X \subseteq V(G) \text{ vtx cover}\}}_{\tau(G)}$$

$\leq$ easy.

**Goal**: find vtx cover of size $\nu(G)$.

**Defn**: Let $M$ be a matching. $P = v_0 v_1 \cdots v_k$ is an <u>$M$-alternating path</u> if $v_0 \notin V(M)$ and the graph induced by $E(P) \setminus E(M)$ is a matching.

**Defn**: $G$ graph, $M \subseteq G$ matching. A path $P = v_0 v_1 \cdots v_k$ is an <u>$M$-augmenting path in $G$</u> if $P$ is $M$-alternating and $\{v_0, v_k\} \cap V(M) = \phi$.

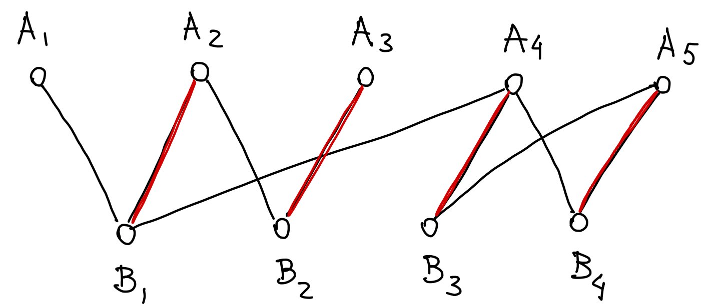
**Obs**: $M$ maximum matching in $G \Rightarrow \nexists$ $M$-augmenting path in $G$.

proof that $\nu(G) \geq \tau(G)$: $(A,B)$ = a bipartition of $G$, $M$ = a max. matching.

Define $f: E(M) \to V(M)$ as $f(ab) = \begin{cases} b, & \text{if } \exists\ x \to b \text{ } M\text{-alternating path, for some } x \in A \\ a, & \text{otherwise.} \end{cases}$

Set $U = f(E(M^*))$.

Example:




$U = \{B_1, B_2, A_4, A_5\}$.

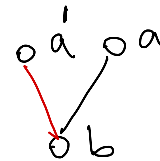
Goal: show that $U$ is a vertex cover.

Let $ab \in E(G) \setminus E(M)$. If $a \in \mathcal{U}$, then we're done.

Assume $a \notin \mathcal{U}$.

<u>Goal</u>: to show that $b \in \mathcal{U}$. That is, $\exists$ $x$-$b$ $M$-alternating path for some $x \in A$.

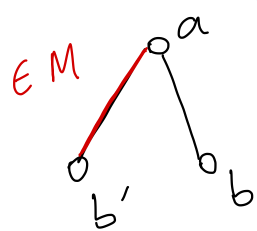
<u>Case 1</u>: $a \notin V(M)$




$\leftarrow$ $M$-alternating, done!

$\Rightarrow b \in V(M)$, oth. we'd have $M$-augmenting

<u>Case 2</u>: $a \in V(M)$.




$a \notin \mathcal{U} \Rightarrow b' \in \mathcal{U}$. $\Rightarrow \exists$ $M$-alternating path ending in $b'$.

$\Rightarrow \exists$ ———————— $b$

$\Rightarrow b \in V(M)$, oth. the $M$-alternating path is $M$-augmenting! $\Rightarrow b \in \mathcal{U}$. $\blacksquare$

Q/ What are the necessary and necessary conditions for $G[A,B]$ to contain a matching covering A?

Defn: M covers A if $A \subseteq V(M)$.

Necessity:

① $|B| \geq |A|$

~~②~~ The size of $N_G(A) := \bigcup_{a \in A} N_G(a)$ is $\geq |A|$.

Note: ② ⇒ ①: $|B| \geq |N_G(A)| \geq |A|$.

② In general we need $|N_G(S)| \geq |S|$.

Thm (Hall '35)

$G[A,B] \supseteq$ matching covering A $\Leftrightarrow \underbrace{|N_G(S)| \geq |S| \quad \forall\, S \subseteq A}_{\text{Hall's condition, or marriage condition}}$