Thm (Hall '35)

$G[A,B] \supseteq$ matching covering $A \iff \underbrace{|N_G(S)| \geq |S| \quad \forall\, S \subseteq A}_{\text{Hall's condition, or marriage condition.}}$

($\Rightarrow$) done!

($\Leftarrow$) Induction on $|A|$. B.I : $|A| = 1$ ✓

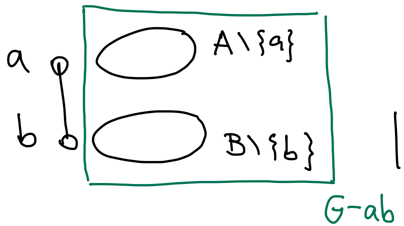
Let $a \geq 2$.

$$\boxed{\begin{array}{l}\text{I.H.}\\ |X| \leq a-1 \;\&\; H[X,Y] \text{ satisfying Hall's cond} \Rightarrow H[X,Y] \supseteq \text{matching covering } X\end{array}}$$

Let $G[A,B]$ be a graph satisfying Hall's cond., $|A| = a$.

Claim : If $|N_G(S)| \geq |S| + 1 \;\; \forall\, \phi \subsetneq S \subsetneq A$, then we're done.

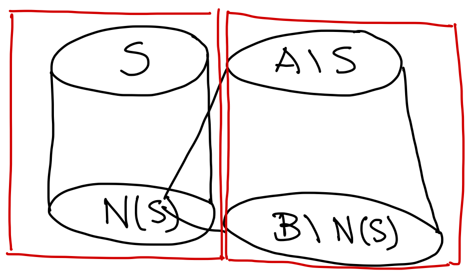
Proof: $ab \in E(G)$.



$G-ab$ satisfies Hall's cond.

$|N_{G-ab}(S)| \geq |N_G(S)| - 1 \geq |S|$ if $S \neq \emptyset$ and $S \neq A$.

Assume $\exists\ \emptyset \subsetneq S \subsetneq A$ such that $|N_G(S)| = |S|$.



$H_1 := G[S, N(S)]$ $\qquad$ $H_2 = G - (S \cup N(S))$.

- I.H. $\Rightarrow \exists$ matching in H_1 covering S.

Claim : H_2 satisfies Hall's condition.

proof : Suppose not. $\exists\, T \subseteq A \setminus S$ such that $|N_G(T) \setminus N_G(S)| \leq |T| - 1$.

$$|N_G(S \cup T)| \leq |N_G(S)| + |N_G(T) \setminus N_G(S)| \leq |S| + |T| - 1 \quad , \text{ contradiction!}$$

Applications

Defn : $M \subseteq G$ is a perfect matching in G or a 1-factor if $V(M) = V(G)$.

Lemma : Any k-regular bipartite graph (with $k \geq 1$) has a perfect matching.

proof : $G[A,B]$ k-regular. For $S \subseteq A$ or $S \subseteq B$ we have

$$k|S| = e(G[S, N(S)]) \leq |N(S)| \cdot k$$

- $S = A \Rightarrow |A| \leq |B|$ and $S = B \Rightarrow |B| \leq |A|$.

  Conclusion: $|A| = |B|$.

- Hall's condition $\Rightarrow \exists$ matching covering A. As $|A| = |B|$, this must be a PM. ∎

Defn: H is a K-factor of G if $H \subseteq G$ is K-regular and $V(H) = V(G)$.

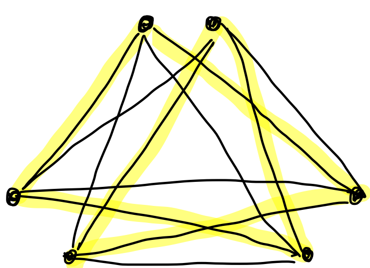
Lemma 2: G $2K$-regular $(K \geq 1) \Rightarrow G$ has a 2-factor.

Proof: G has an Eulerian tour. Let D be the associated digraph.

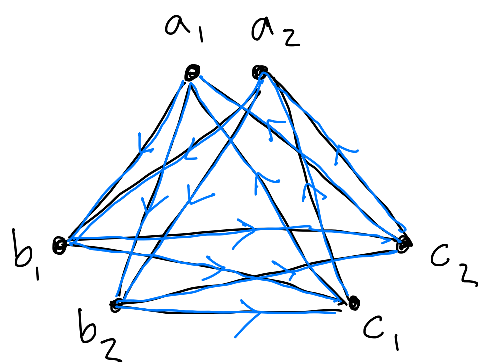
Split each $v \in V(D)$ into v^- and v^+.

Define H as $V(H) = \{v^+, v^- : v \in V(D)\}$ and $E(H) = \{u^+v^- : (u,v) \in E(D)\}$

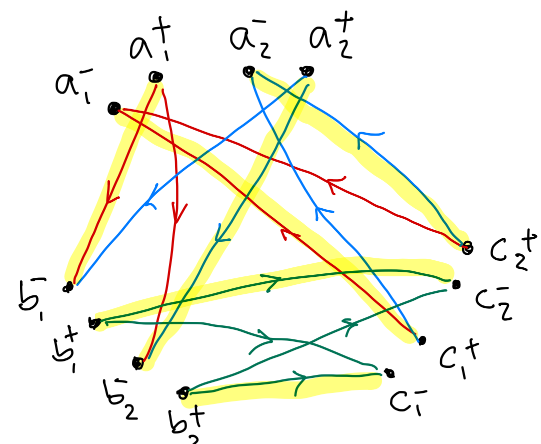
Example:



G



D

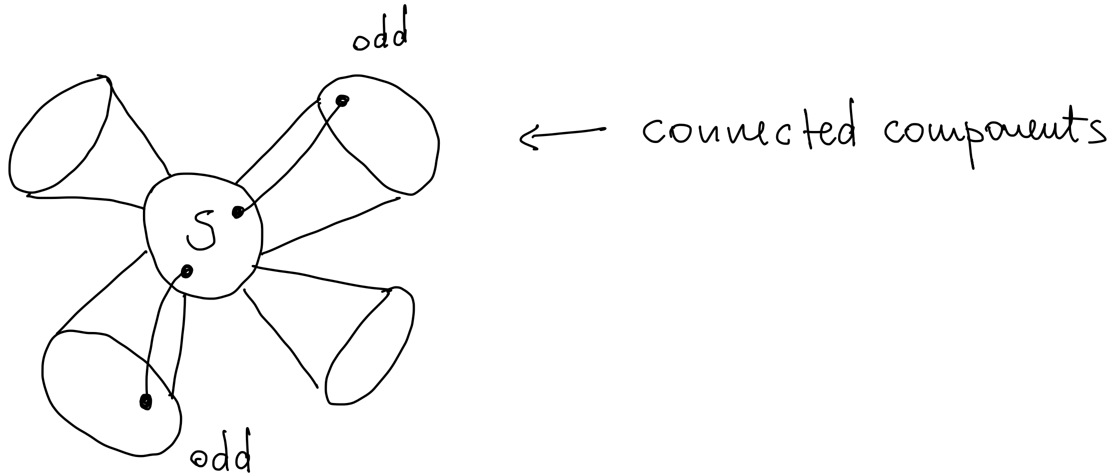


H

H is K-regular and bipartite! By Lemma 1, H has P.M. M.

As $d_M(v^-) = d_M(v^+) = 1$, mapping M back to G gives a 2-factor. $\blacksquare$

Q/ When does a graph have a P.M.?



Notation: $q(H) :=$ # connected components with an odd # of vtxs.

Thm (Tutte) Let G be a connected graph.

$$G \text{ has a P.M} \iff q(G-S) \leq |S| \ \forall S.$$

Defn: C is critical if $C-v$ has a P.M. $\forall v \in V(C)$.

Notation: $\mathcal{C}_H :=$ set of all connected components of H.

Defn: $S \subseteq G$ is matchable with $\mathcal{C}_{G-S}$ if the auxiliary graph H_S with $V(H_S) = S \cup \mathcal{C}_{G-S}$ and $E(H_S) = \{ sC : s \in S \text{ and } N_G(s) \cap V(C) \neq \emptyset \}$ has a matching covering S.

Gallai-decomposition thm: Let G be a connected graph. $\exists S^* \subseteq V(G)$ such that

1. $C \in \mathcal{C}_{G-S^*} \Rightarrow C$ is critical;
2. S^* is matchable with $\mathcal{C}_{G-S^*}$.

Moreover, G has a P.M. $\iff |\mathcal{C}_{G-S^*}| = |S^*|$.