Thm (Tutte) Let G be a connected graph.

$$G \text{ has a P.M} \Leftrightarrow q(G-S) \leq |S| \ \forall S.$$

Defn: C is critical if $C-v$ has a P.M. $\forall v \in V(C)$.

Notation: $\mathcal{C}_H :=$ set of all connected components of H.

Defn: $S \subseteq G$ is matchable with $\mathcal{C}_{G-S}$ if the auxiliary graph H_S with $V(H_S) = S \cup \mathcal{C}_{G-S}$ and $E(H_S) = \{ sC : s \in S \text{ and } N_G(s) \cap V(C) \neq \phi \}$ has a matching covering S.

Gallai-decomposition thm: Let G be a connected graph. $\exists\, S^* \subseteq V(G)$ such that

1. $C \in \mathcal{C}_{G-S^*} \Rightarrow C$ is critical ;
2. S^* is matchable with $\mathcal{C}_{G-S^*}$.

Moreover, G has a P.M. $\Leftrightarrow |\mathcal{C}_{G-S^*}| = |S^*|$.

$|\mathcal{C}_{G-S^*}| = |S^*| \Rightarrow G$ has P.M.

G has P.M $\Rightarrow |\mathcal{C}_{G-S^*}| = q(G-S^*) \leq |S^*|$.

S^* matchable $\Rightarrow |\mathcal{C}_{G-S^*}| \geq |S^*|$.

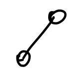
<u>proof of Tutte's thm, assuming G.D</u>: Suppose that $q(G-S) \leq |S| \ \forall \ S \subseteq V(G)$.

S^* given by G.D. <u>Goal</u>: to show that $|\mathcal{C}_{G-S^*}| = |S^*|$.

① $\Rightarrow |C|$ is odd $\forall \ C \in \mathcal{C}_{G-S^*}$

$\Rightarrow |\mathcal{C}_{G-S^*}| = q(G-S^*) \leq |S^*|$.

② $\Rightarrow \exists$ matching in H_{S^*} covering $S^* \Rightarrow |S^*| \leq |\mathcal{C}_{G-S^*}|$.

<u>Proof of G.D</u>: induction on # vtxs. B.I: 

Assume G.D. thm holds for any H with $v(H) < v(G)$. In particular, <u>Tutte's thm holds for any H with $v(H) < v(G)$.</u>

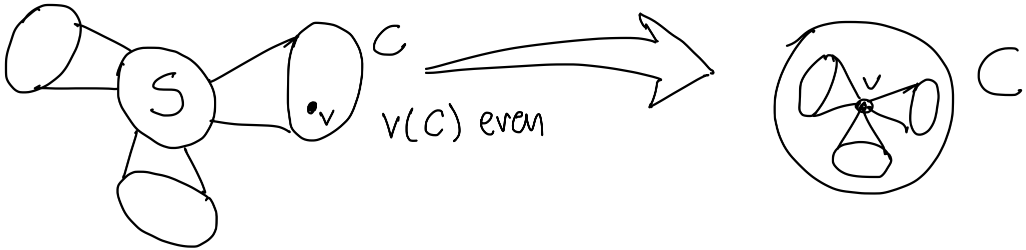
Define $r(T) := q(G-T) - |T| \ \forall T$.

Let $\mathcal{R} = \{T : r(T) = \max_{T'} r(T')\}$ and S be so that $|S| = \max\{|T| : T \in \mathcal{R}\}$

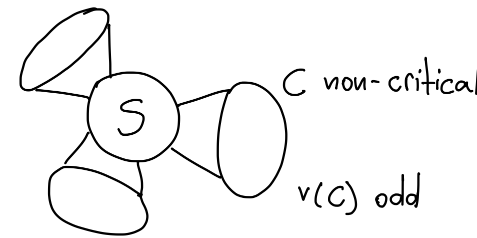
<u>Claim</u>: $v(C)$ is odd $\forall \ C \in \mathcal{C}_{G-S}$. In particular, $|\mathcal{C}_{G-S}| = q(G-S)$.

<u>proof</u>: Suppose not and let $C \in \mathcal{C}_{G-S}$ with $|C|$ even. Let $v \in V(C)$.

$v(C)$ even $\Rightarrow q(C-v) \geq 1$. Then,



$q(G-S-v) \geq q(G-S)+1 \Rightarrow r(S \cup \{v\}) \geq r(S)$, contradicting the choice of S



<u>proof of ①</u>: Suppose $\exists$ non-critical $C \in \mathcal{C}_{G-S}$.

$\exists\ v \in V(C)$ such that $C-v \not\supseteq$ P.M.

<u>Induction hypothesis</u> $\Rightarrow \exists\ S' \subseteq V(C-v)$ st $q(C-v-S') > |S'|$.

$q(C-v-S') + |S'| \equiv v(C-v) \equiv 0 \pmod 2 \Rightarrow q(C-v-S') \geq |S'|+2$.

<u>Claim</u>: $r(S \cup S' \cup \{v\}) \geq r(S)$. (this is a contradiction!)

proof: $q(G-S-S'-v) \geq q(G-S) - 1 + q(C-S'-v)$

$$\geq q(G-S) + |S'| + 1.$$

$\Rightarrow r(S \cup S' \cup \{v\}) \geq r(S)$ ■

proof of ②: Suppose by contradiction that S is not matchable.

Hall's thm $\Rightarrow |N_H(S')| < |S'|$ for some $S' \subseteq S$.

Recall claim.

Claim: $r(S \setminus S') > r(S)$ (this is a contradiction!)

proof: $\mathcal{C}_{G-S} \setminus N_{H_S}(S')$ are also odd connected components of $G-(S\setminus S')$.

$q(G-(S\setminus S')) \geq q(G-S) - |N_{H_S}(S')| > q(G-S) - |S'| \Rightarrow r(S\setminus S') > r(S)$ ■

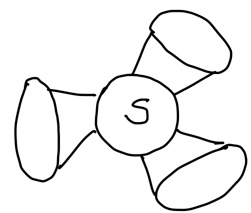
Application

Defn: G is ℓ-edge-connected if $\forall\, F \subseteq E(G)$ with $|F| \leq \ell - 1$ we have that $G - F$ is connected.

Lemma: G 3-regular and 2-edge-connected $\Rightarrow$ G has a P.M.

proof: Goal: to show that $|S| \geq q(G-S)$.

$\sigma = \{ C \in \mathcal{C}_{G-S} : v(C) \text{ is odd} \}$



$e_G(S,C) = \#\{ e \in E(G) : |e \cap S| = |e \cap C| = 1 \}$

$$3|S| \geq \sum_{C \in \sigma} e_G(S,C) \geq 3|\sigma| = 3q(G-S).$$

Claim: $e_G(S,C) \geq 3 \;\; \forall\, C \in \sigma$.

proof: $3v(C) = \sum_{v \in V(C)} d(v) = 2e(G[C]) + e_G(S,C) \Rightarrow e_G(S,C)$ is odd.

G 2-edge-connected $\Rightarrow e_G(S,C) \neq 1$ ▤

Notation 1: $K[A,B]$ = the complete bipartite graph with parts A and B.

Defn: G is a $(v_1,\dots,v_r)$-rooted layered graph if $\exists$ disjoint $N_0,\dots,N_h \subseteq V(G)$ such that $N_0=\{v_1,\dots,v_r\}$, $V(G)=N_0\cup\dots\cup N_h$ and $E(G)\subseteq \bigcup_{i\in[h]} E(K[N_{i-1},N_i])$

We call N_i the ith level of G.

h = height of G and $L(G):=N_h$ is the last level of G.

Given a matching $M\subseteq G$, we say that G is M-alternating if

$$E(G[N_i\cup N_{i+1}])\subseteq \begin{cases} E(G)\setminus E(M) & \text{if } i\equiv 0 \bmod 2 \\ E(M) & \text{oth.}\end{cases}$$

Notation 2: $A\,\Delta\, B = (A\setminus B)\cup(B\setminus A)$

The Hopcroft-Karp Algorithm

Input: a bipartite graph $G[A,B]$.

Set $M_0 := \phi$ and $i := 0$

While M_i is not maximum matching, do:

① Set $s := |A \setminus V(M_i)|$ and label vtxs in $A \setminus V(M_i)$ as $\{v_1, \dots, v_s\}$.

② Use B.F.S to build a $(v_1, \dots, v_s)$-rooted layered M_i-alternating graph T_i

Stop building T_i when $L(T_i) \setminus V(M_i) \neq \phi$.

③ Find a maximal collection $\mathcal{P}_i$ of vtx-disjoint shortest M_i-augmenting paths in T_i

④ Set $M_{i+1} = M_i \, \Delta \, \underbrace{P^{(1)} \, \Delta \cdots \Delta \, P^{(r)}}_{\forall P^{(j)} \in \mathcal{P}_i}$.

Output: M_i