Notation 1: $K[A,B]$ = the complete bipartite graph with parts A and B.

Defn: G is a $(v_1,\dots,v_r)$-**rooted layered graph** if $\exists$ disjoint $N_0,\dots,N_h \subseteq V(G)$ such that $N_0 = \{v_1,\dots,v_r\}$, $V(G) = N_0 \cup \dots \cup N_h$ and $E(G) \subseteq \bigcup_{i \in [h]} E(K[N_{i-1}, N_i])$

We call N_i the ith level of G.

h = **height of G** and $L(G) := N_h$ is **the last level of G**.

Given a matching $M \subseteq G$, we say that G is **M-alternating** if

$$E(G[N_i \cup N_{i+1}]) \subseteq \begin{cases} E(G) \setminus E(M) & \text{if } i \equiv 0 \bmod 2 \\ E(M) & \text{oth.} \end{cases}$$

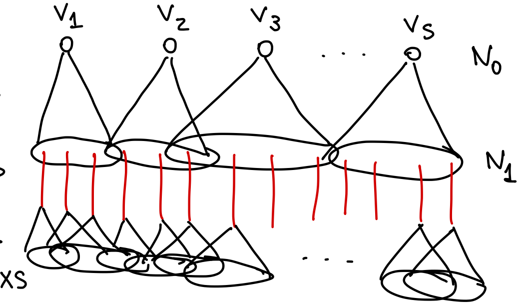
Notation 2: $A \Delta B = (A \setminus B) \cup (B \setminus A)$

The Hopcroft-Karp Algorithm

Input: a bipartite graph $G[A,B]$.

Set $M_0 := \emptyset$ and $i := 0$

While M_i is not maximum matching, do:



① Set $s := |A \setminus V(M_i)|$ and label vtxs in $A \setminus V(M_i)$ as $\{v_1, \dots, v_s\}$.

② Use B.F.S to build a $(v_1, \dots, v_s)$-rooted layered M_i-alternating graph T_i.
Stop building T_i when $L(T_i) \setminus V(M_i) \neq \emptyset$.

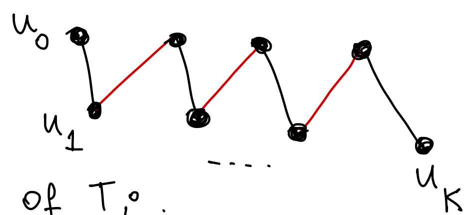
③ Find a maximal collection $\mathcal{P}_i$ of vtx-disjoint shortest M_i-augmenting paths in T_i

④ Set $M_{i+1} = M_i \,\Delta\, \underbrace{P^{(1)} \,\Delta \cdots \Delta\, P^{(r)}}_{\forall P^{(j)} \in \mathcal{P}_i}$.

Output: M_i

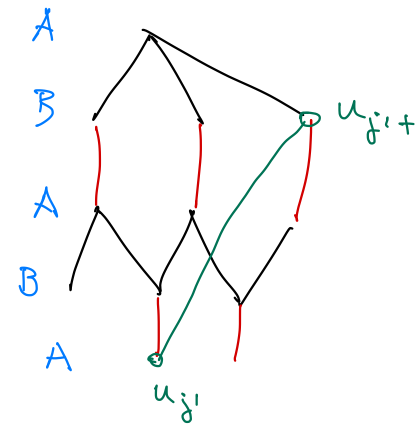
Lemma1: M is a maximum matching in $G \iff \nexists$ M-augmenting paths.
(exercise).

Lemma2: Let P be a shortest M_i-augmenting path in G. Then, $P \subseteq T_i$.

proof: $P =$ 

WLOG, assume $u_0 \in A$ and $u_K \in B$.

$N_j :=$ the j-th level of T_i.



Claim: $u_{j'} \in N_{\leq j'} := N_0 \cup \cdots \cup N_{j'}$ $\forall j' \in \{0, 1, \ldots, K\}$.

proof: Induction on j'.

$u_{j'} \in N_{\leq j'} \Rightarrow u_{j'+1} \in N(u_{j'}) \subseteq N_{\leq j'+1}$ by B.F.S.

Obs: T_i has $\geq K$ levels, as every unmatched vtx in B is in $L(G)$

Claim $\Rightarrow V(P) \subseteq T_i$ and $u_K \in L(T_i)$, as the only unmatched vtxs are in $L(T_i)$.

In particular, the height of T_i is K.

Goal: $u_j \in N_j \; \forall j \in \{0, \dots, K\}$. (as this implies $u_{j-1} u_j \in T_i[N_{j-1}, N_j]$)

Define $f : \{0, \dots, K\} \to \{0, \dots, K\}$ as $f(j) = i$ iff $u_j \in N_i$.

Obs: $f(j) > f(j-1) \Rightarrow f(j) = f(j-1) + 1$.

Claim + Obs $\Rightarrow$

$$K = f(K) - f(0) = \sum_{j=1}^{K} f(j) - f(j-1) \leq K$$

Equality occurs $\Leftrightarrow f(j) = f(j-1)+1$.

$$\Downarrow$$

$$f(j) = j \; \forall j.$$

Corollary: the algorithm works correctly.

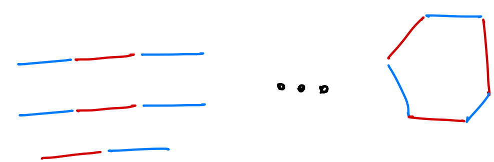
proof: $|M_i| < |M_{i+1}| \leq v(G)/2$ until $\nexists$ M_i-augmenting path. $\blacksquare$

Lemma 3: Let $M \subseteq G$ be a matching and $P \subseteq G$ be a shortest M-augmenting path in G. If Q is an $M \Delta P$-augmenting path in G, then

$$e(Q) \geq e(P) + e(P \cap Q).$$

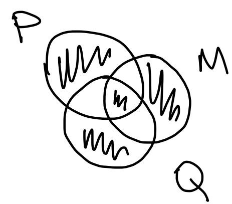
proof: P M-augmenting $\Rightarrow |M \Delta P| \geq |M| + 1$.

Q $M\Delta P$-augmenting $\Rightarrow |\underbrace{(M \Delta P) \Delta Q}_{:= N}| \geq |M \Delta P| + 1 \geq |M| + 2$.

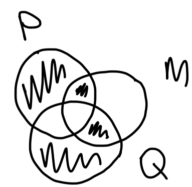
$M =$  and $N =$ . $N \Delta M =$  alternating paths and cycles.

$|N| \geq |M|+2 \Rightarrow \exists \geq 2$ paths, say P_1 and P_2, whose start and end edge is blue.

$\Rightarrow P_1$ and P_2 are M-augmenting $\Rightarrow e(P_1), e(P_2) \geq e(P)$.



$\Rightarrow$



$M \Delta P \Delta Q = N$

$\boxed{M \Delta N = P \Delta Q}$

Another proof: Δ is associative and $M \Delta M = \phi$.

$$e(P) + e(Q) - e(P \cap Q) = e(P \Delta Q) = e(M \Delta N) \geq e(P_1) + e(P_2) \geq 2e(P)$$

$$\Rightarrow e(Q) \geq e(P) + e(P \cap Q).$$