Lemma 1: M is a maximum matching in $G \iff \nexists$ M-augmenting paths. (exercise).

Lemma 2: Let P be a shortest M_i-augmenting path in G. Then, $P \subseteq T_i$.

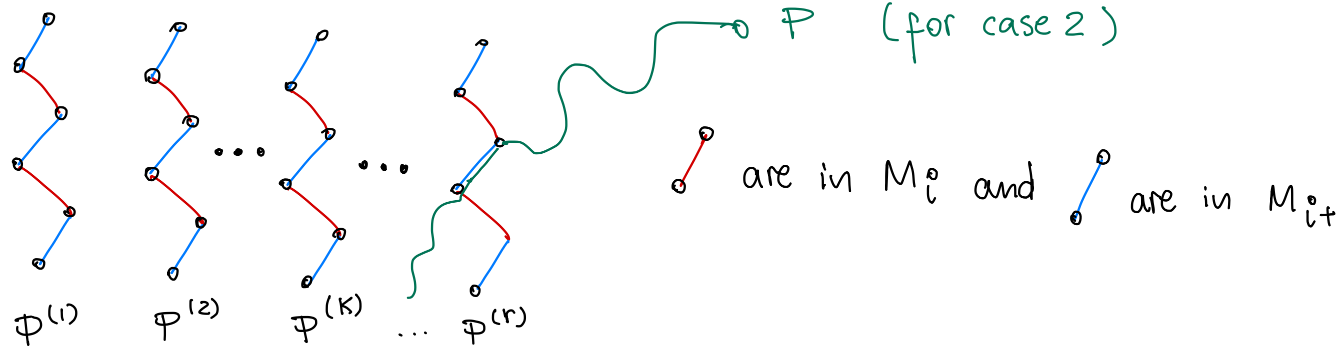
Lemma 3: Let $M \subseteq G$ be a matching and $P \subseteq G$ be a shortest M-augmenting path in G. If Q is an $M \Delta P$-augmenting path in G, then

$$e(Q) \geq e(P) + e(P \cap Q).$$

Lemma: Let l_i be the length of the shortest M_i-augmenting path in G. Then, $l_{i+1} \geq l_i + 1$.

proof: By Lemma 3, we have $\ell_{i+1} \geq \ell_i$.

$\mathcal{P}_i =$



Let P be a M_{i+1}-augmenting path in G. Goal: to show that $e(P) \geq \ell_i + 1$.

- Case 1: $V(P) \cap V(P^{(j)}) = \phi \;\; \forall j \Rightarrow P$ is a M_i-augmenting path in G

$$\Rightarrow e(P) \geq \ell_i + 1$$

why: $\mathcal{P}_i$ is a maximal collection of shortest M_i-augmenting paths in G.

- Case 2: $V(P) \cap V(P^{(j)}) \neq \emptyset$ for some j.

Let $K = \max\{ j : V(P) \cap V(P^{(j)}) \neq \emptyset \}$.

Obs1: $P^{(K)}$ is a shortest $\underbrace{M_i \Delta P^{(1)} \Delta \cdots \Delta P^{(K-1)}}_{:= N}$ - augmenting path in G.

By L3, taking these symmetric differences never decrease the length of the shortest augmenting path.

Obs2: P is a $N \Delta P^{(K)}$ - augmenting path in G.

By L3, $e(P) \geq e(P^{(K)}) + e(P \cap P^{(K)})$.

As P is M_{i+1}-augmenting and M_{i+1} contains the blue edges above, we have $e(P \cap P^{(K)}) \geq 1$. Hence, $e(P) \geq \ell_i + 1$. ∎

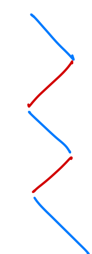
Claim 1: Let M^* be the matching output by the HK algorithm.
Then, $e(P) \geq i$ $\forall$ M_i-augmenting path P in $M^* \Delta M_i$.
In particular, # M_i-augmenting paths in $M^* \Delta M_i \leq n/i$.

proof: The components of $M^* \Delta M_i$ are vtx-disjoint paths and cycles. ☒

Claim 2: $|M^*| - |M_i| \leq \frac{n}{i}$

proof: It suffices to count the edges of M_* and M_i in $M_* \Delta M_i$.

- M_*-augmenting paths don't exist in $M_* \Delta M$.
- P M_i-augmenting path in $M_* \Delta M \Rightarrow e(P \cap M_*) - e(P \cap M_i) = 1$



$\Rightarrow |M^*| - |M_i| \leq$ # M_i-augmenting paths in $M_* \Delta M \leq \frac{n}{i}$ ☒

Running time: $O\left(m \cdot \left(\overset{\sqrt{n}}{\cancel{i + \frac{n}{i}}}\right)\right)$

Connectivity

Defn: G is k-connected if $v(G) \geq k+1$ and $G - X$ is connected $\forall X$ of size $\leq k-1$.

$\kappa(G) := \max\{k : G \text{ is } k\text{-connected}\}$ is called the connectivity of G.

Defn: $X \subseteq V(G)$ is a separator if $G - X$ is disconnected.

Obs: $\delta(G) \geq \kappa(G)$. ← It doesn't say much about the connectivity of G. But we do have a "highly" connected subgraph.

Thm (Mader '72) Let $k \in \mathbb{N}_{\geq 1}$ and G be a graph with average degree $d(G) \geq 4k$.
Then, G contains a $(k+1)$-connected subgraph H with $d(H) \geq d(G) - 2k + 1$

proof: Let $d^* := d(G)$ and let $H \subseteq G$ be so that

$$e(H) = \min\left\{ e(G') : G' \subseteq G,\ v(G') \geq 2k \text{ and } e(G') \geq \frac{d^*}{2}(v(G') - k) + 1 \right\}$$

↑ Note that G is in this set. Hence the minimum is finite.

Goal: to show that H is $(K+1)$-connected and $d(H) \geq d(G) - 2K$.

Claim 1: $v(H) \geq 2K+1$

proof: $e(H) \geq \frac{d^*}{2}(v(H) - K) \geq 2K \cdot (2K - K) = 2K^2$

Suppose for contradiction that $v(H) \leq 2K$. Then, $e(H) \leq e(K_{2K})$

$$= \binom{2K}{2} = K \cdot (2K-1)$$

contradiction! $\blacksquare$

Claim 2: $\delta(H) \geq \frac{d^*}{2} + 1$

proof: Suppose for contradiction that $\exists\, u \in V(H)$ such that $d_H(u) \leq \frac{d^*}{2}$.

Claim 1 $\Rightarrow$ $v(H-u) \geq 2K$. As $e(H-u) \geq e(H) - \frac{d^*}{2} \geq \frac{d^*}{2} \cdot (v(H) - 1 - K)$

$$\geq \frac{d^*}{2}(v(H-u) - K).$$

This contradicts the choice of H. $\blacksquare$

Claim 3: $d(H) > d(G) - 2K$.

proof: $$d(H) = \frac{2e(H)}{v(H)} > \frac{d^*(v(H)-K)}{v(H)} = d^* - \frac{d^* K}{v(H)} \geqslant d^* - 2K$$

AS $v(H) \geq d^*/2$.

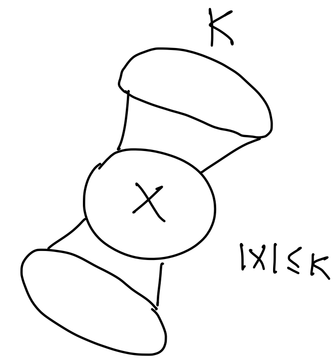
$\blacksquare$

Suppose by contradiction that H is not $(K+1)$-connected.

$\exists\, X \subseteq V(H)$, $|X| \leqslant K$, such that $H - X$ is disconnected.

Let K be a component of $H - X$.

$H_1 = H[K \cup X]$ and $H_2 = H - K$.



Claim 4: For $i \in \{1,2\}$ $\exists\, u \in V(H_i)$ such that $N_{H_i}(u) = N_H(u)$.

proof: Let C be a connected component of $H - X$. Then,

$$v \in V(C) \Rightarrow N_H(v) \subseteq V(C) \cup X \qquad \blacksquare$$

Claims 2+4 $\Rightarrow$ $v(H_1), v(H_2) \geq \frac{d^*}{2} = 2K$.

By the minimality of H, we have $e(H_i) \leq \frac{d^*}{2}(v(H_i) - K)$ for $i \in \{1,2\}$

Thus, $$e(H) \leq e(H_1) + e(H_2) \leq \frac{d^*}{2} \cdot (v(H_1) + v(H_2) - 2K)$$

$$= \frac{d^*}{2} \cdot (v(H) + |X| - 2K)$$

$$\leq \frac{d^*}{2}(v(H) - K)$$

This contradicts the choice of H. $\blacksquare$