Defn: Let $x, y \in V(G)$ and $U \subseteq V(G) \setminus \{x,y\}$. We say that U separates x and y, or that U is a x-y separator if every x-y path in G contains a vertex of U.

Defn: A collection $\{P^{(1)}, \dots, P^{(r)}\}$ of x-y paths are (internally) vtx-disjoint if $V(P^{(i)}) \cap V(P^{(j)}) = \{x,y\} \ \forall\, i \neq j$.

~~Obs~~ **Menger's thm**: Let $xy \in E(\overline{G})$. Then,

$$\min\{|U| : U \text{ is a } x\text{-}y \text{ separator in } G\} \cancel{\geq} \max\left\{|\mathcal{P}| : \mathcal{P} \text{ collection of vtx-disjoint } x\text{-}y \text{ paths in } G\right\}$$

proof: Induction on $e(G)$. Basis of induction: the empty graph.

Let $xy \in E(\overline{G})$ and $K := \min\{|U| : U \text{ is a } x\text{-}y \text{ separator in } G\}$

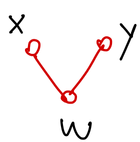
Goal: show that $\underbrace{\exists \text{ collection } \mathcal{P} \text{ of } k \text{ vtx-disjoint } x\text{-}y \text{ paths in } G}_{(*)}$.

$(*)$ is true when $k \leq 1$. Thus, assume $k \geq 2$.

Case 1: $\exists\ w \in N_G(x) \cap N_G(y)$.

U x-y separator in $G - w \Rightarrow |U| \geq k-1$.

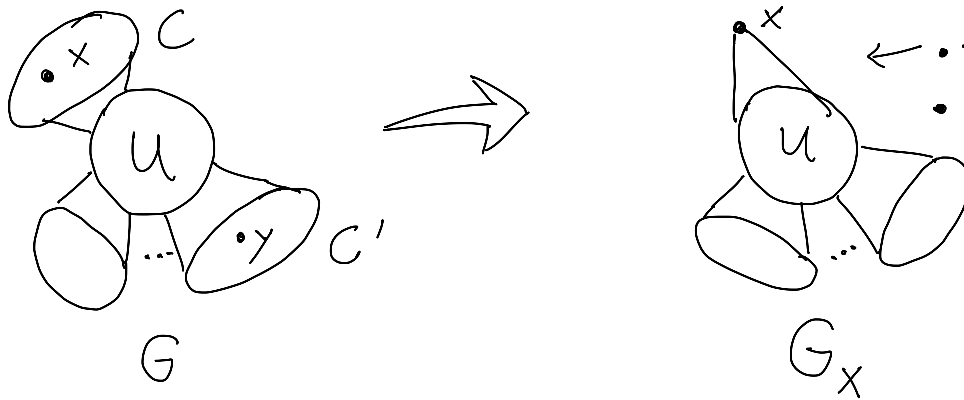
I.H. $\Rightarrow \exists \geq k-1$ vtx-disjoint x-y paths in $G - w$.

$+$ 

$\Rightarrow \exists \geq k$ ———————————————— G. ✓

Case 2: $\exists$ a x-y separator U in G, $|U| = k$, such that

$U \not\subseteq N_G(x)$ and $U \not\subseteq N_G(y)$.



- join x to all $u \in \mathcal{U}$.
- delete $C-x$.

<u>Claim</u>: $e(G_x) < e(G)$.

<u>proof</u>: • $N_G(x) \cap C \neq \phi$, and hence $e(C) \geq 1$.

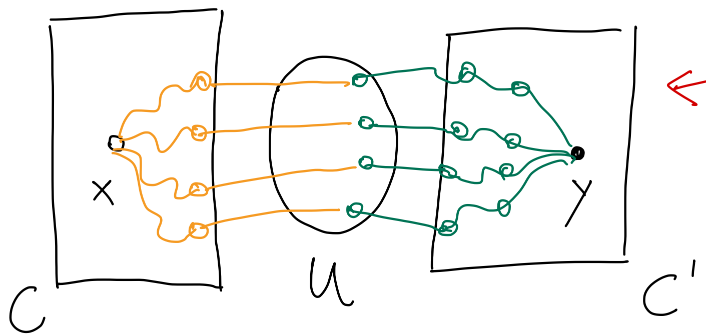
[Otherwise, $N_G(x) \subseteq \mathcal{U} \Rightarrow N_G(x) = \mathcal{U}$, as $N_G(x)$ is a x-y sep.]

• $N_G(u) \cap V(C) \neq \phi \ \forall u \in \mathcal{U}$.

[$N_G(u) \cap V(C) = \phi \Rightarrow \mathcal{U} \setminus \{u\}$ is a smaller x-y separator]

<u>Obs</u> : U' x-y separator in G_x $\Rightarrow$ $|U'| \geq k$.

I.H + Obs $\Rightarrow$ G_x contains k vtx-disjoint x-y paths.

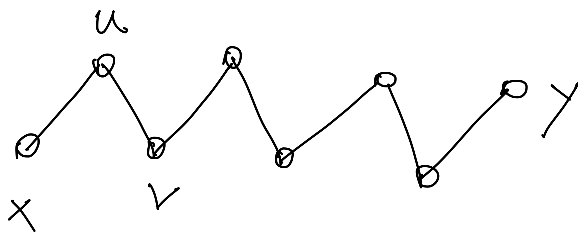


k vtx-disjoint paths from y to U in $G_x[C' \cup U]$, and hence in $G[C' \cup U]$.

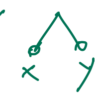
By symmetry, $\exists$ k vtx-disjoint paths from x to U in $G[C \cup U]$.

We can concatenate them to obtain a collection of k vtx-disjoint x-y paths in G.

<u>Case 3</u>: $N_G(x) \cap N_G(y) = \emptyset$ and $\forall$ x-y separator U in G st $|U| = k$, $U \subseteq N_G(x)$ or $U \subseteq N_G(y)$.

$P =$ 

x-y path of minimum length.

u and v exist because $\nexists$ 

.

<u>Obs</u>: $v \notin N_G(x)$ and $v \neq y$

Let $H = G - uv$

- If every x-y separator in H has size $\geq k$, then by the I.H. H contains k vtx-disjoint x-y paths, and so does G.

- Assume $\exists$ x-y separator W in H with $|W| = k-1$.

<u>Claim</u>: $W \subseteq N_G(x) \cap N_G(y)$. (this is a contradiction, as $|W| \geq k-1 \geq 1$)

<u>proof</u>: Let $W_u := W \cup \{u\}$ and $W_v = W \cup \{v\}$.

Both W_u and W_v are x-y separators.

- $W_v \not\subseteq N_G(x)$, as $xv \notin E(G)$

$\Rightarrow W_v \subseteq N_G(y) \Rightarrow W \subseteq N_G(y)$.

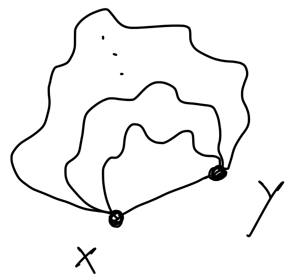
- $W_u \not\subseteq N_G(y)$, as $uy \notin E(G)$.

$\Rightarrow W_u \subseteq N_G(x) \Rightarrow W \subseteq N_G(x)$.

$\blacksquare$

<u>Corollary</u> : G is K- connected $\Leftrightarrow \forall\, x, y \in V(G)$ distinct $\exists\, K$ vtx-disjoint x-y paths.

<u>proof</u> : $(\Leftarrow)$ $K \geq 1$



$\leftarrow K-1$

$\Rightarrow \quad v(G) \geq K-1+2$.

U st $|U| \leq K-1 \Rightarrow v(G-U) \geq 2$.

K- connectedness of $G \Rightarrow \forall\, x, y \in V(G-U)$ distinct $\exists$ a x-y path in $G-U$.

$(\Rightarrow)$ Let $x, y \in V(G)$ distinct.

G K-connected $\Rightarrow$ $G-R$ contains a x-y path $\forall$ $R \subseteq V(G) \setminus \{x,y\}$ of size $|R| = K-1$. ($G-R$ is connected)

If U is a x-y separator, then $|U| \geq K$.

- Case 1 : $xy \notin E(G)$.

  By Menger's thm, $\exists$ K vtx-disjoint x-y paths in G.

- Case 2 : $xy \in E(G)$.

  $G-xy$ is $K-1$ connected. By Menger's thm, $\exists$ $K-1$ vtx-disjoint x-y paths in $G-xy$

  The edge xy gives the Kth x-y path in G.

$\blacksquare$

A generalisation of Menger's theorem.

Defn: For sets S, T, we say that a path $P = v_0 \cdots v_k$ is a S-T path if $v_0 \in S$ and $v_k \in T$.

Defn: We say that $\{P^{(1)}, \ldots, P^{(r)}\}$ is a collection of (internally) vertex-disjoint S-T paths if

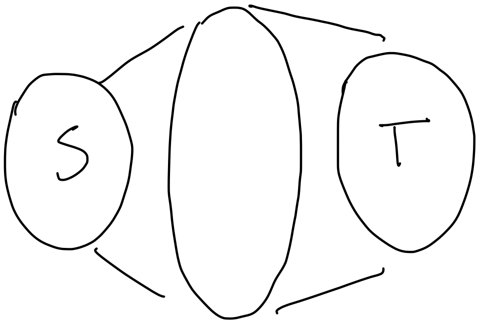
- $P^{(i)}$ is a S-T path $\forall i$
- $V(P^{(i)}) \cap V(P^{(j)}) \subseteq S \cup T \quad \forall i \neq j$.

Defn: U is an S-T separator in G if $G - U$ doesn't have an S-T path and $U \subseteq V(G) \setminus (S \cup T)$.

Thm: $S, T \subseteq V(G)$ disjoint. If $e(G[S,T]) = 0$, then

$$\min\{|U| : U \text{ } S,T\text{-separator}\} = \max\left\{|\mathcal{P}| : \mathcal{P} \text{ collection of vtx-disjoint } S\text{-}T \text{ paths in } G.\right\}$$

proof:



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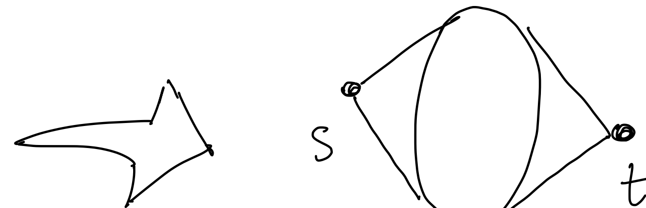

G'

Replace S by s, T by t

Set $N_{G'}(s) := \bigcup_{x \in S} N_G(x)$ and $N_{G'}(t) := \bigcup_{y \in T} N_G(y)$

U is S-T separator in $G \iff U$ is a s-t separator in G'.

Proof follows from Menger's theorem. $\blacksquare$