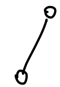
Lemma: $e(G) \geq 1 \Rightarrow \exists H \subseteq G$ st $\delta(H) > \frac{e(H)}{v(H)} \geq \frac{e(G)}{v(G)}$.

proof: Induction on $v(G)$. BI: 

Assume $\delta(G) \leq \frac{e(G)}{v(G)}$ and set $n := v(G)$.

$$\boxed{\text{Claim: If } d(x) \leq \frac{e(G)}{n}, \text{ then } \frac{e(G-x)}{n-1} \geq \frac{e(G)}{n}.}$$

proof:

$$e(G-x) = e(G) - d(x) \geq e(G) \cdot \left(1 - \frac{1}{n}\right)$$

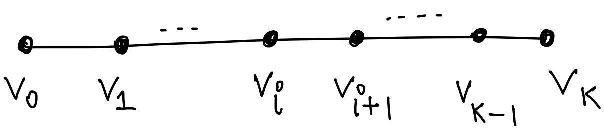
$$= e(G) \cdot \frac{n-1}{n}$$

$\blacksquare$

Lemma: $P_{\delta(G)} \subseteq G$ and $C_\ell \subseteq G$ for some $\ell \geq \delta(G)+1$.

proof:

Longest path:



$$N(v_0) \subseteq \{v_1, \dots, v_k\} \Rightarrow k \geq d(v_0) \geq \delta(G).$$

$$j := \max\{i : v_i \sim v_0\}.$$

$$N(v_0) \subseteq \{v_1, \dots, v_j\} \Rightarrow C_{j+1} \subseteq G \text{ and } j \geq \delta(G).$$

$\blacksquare$

Corollary: $\delta(G) \geq 2 \Rightarrow G$ has a cycle.

<u>Defn</u>: A walk is an alternating sequence of vtxs and edges $v_1 e_1 v_2 e_2 \cdots v_k e_k v_{k+1}$ such that $e_i = v_i v_{i+1}$ $\forall i$.

k ← length of the walk.

<u>x-y walk</u>: A walk with endpoints x and y.

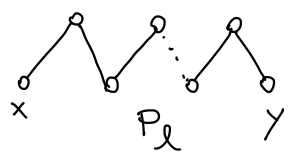
<u>x-y path</u>: A path ____________________________.

<u>Prop</u>: G graph, $x, y \in V(G)$.

$$\exists \text{ } x\text{-}y \text{ walk in } G \iff \exists \text{ } x\text{-}y \text{ path in } G.$$

proof: the shortest x-y walk is a path. $\blacksquare$

Corollary: If $\exists$ x-y path and y-z path in G, then $\exists$ x-z path in G.

Distance: $d_G(x,y) = \min\{ \ell :$ 

$\}$

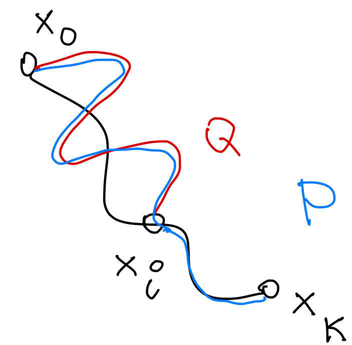
If $\nexists$ such ℓ, set $d_G(x,y) = \infty$.

Diameter: $\max\limits_{x,y} d_G(x,y)$.

<u>Proposition 1:</u> G graph and $G \supseteq P = x_0 \cdots x_k$ path.

If $d_G(x_0, x_k) = k$, then $d_G(x_0, x_i) = i = d_G(x_k, x_{k-i})$ $\forall i$.

<u>proof</u>:



- We have $d_G(x_0, x_i) \leq i$.
- Let Q be a $x_0 - x_i$ path
- $Q \cup x_i P x_k$ contains a $x_0 - x_k$ path P'

$\Rightarrow$ $e(Q) + k - i \geq |P'| \geq d_G(x_0, x_k) = k$

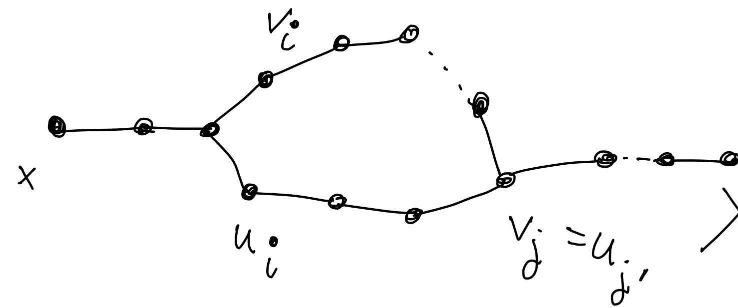
$\Rightarrow$ $e(Q) \geq i$

$\Rightarrow$ $d_G(x_0, x_i) = i$

$\blacksquare$

Proposition 2: If $\exists$ distinct paths P and Q in G with same endpoints, then G contains a cycle.

proof: x, y endpoints



P: $x = v_0 v_1 \cdots v_k = y$

Q: $x = u_0 u_1 \cdots u_\ell = y$

$i := \min\{t : v_t \neq u_t\}$

$j := \min\{t > i : v_t \in Q\}$

j' be so that $v_j = u_{j'}$ $(j' > i)$

$v_{i-1} P v_j \cup u_{i-1} Q u_{j'}$ is a cycle! $\blacksquare$

<u>Girth</u>: $g(G) := \min\{\ell \geq 3 : C_\ell \subseteq G\}$

If $\nexists$ such ℓ, $g(G) = \infty$.

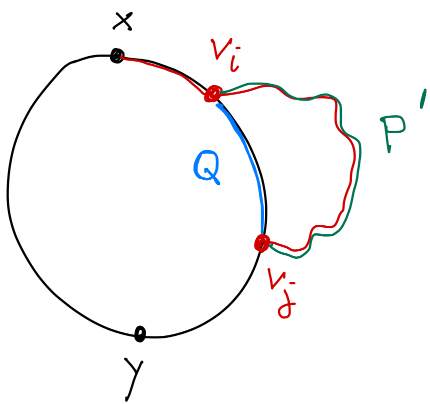
<u>Lemma</u>: If G contains a cycle, then $g(G) \leq 2\,\mathrm{diam}(G) + 1$.

<u>proof</u>: Suppose not.

- Let C be a shortest cycle. Then,

$$e(C) = v(C) \geq 2\,\mathrm{diam}(G) + 2$$

- $\exists\, x, y \in V(C)$ st $d_C(x,y) \geq \mathrm{diam}(G) + 1$.



- $\exists$ path $P \subseteq G$ and $P \nsubseteq C$ with endpoints x, y st $e(P) \leqslant \operatorname{diam}(G)$

- $x = v_0 \, v_1 \cdots v_k = y$ vtxs of P.

- $i = \min\{t : v_t \notin V(C)\}$ and $j = \min\{t > i : v_t \in V(C)\}$

$Q \cup P'$ is a cycle of length $\leqslant \left\lfloor \frac{v(C)}{2} \right\rfloor + \operatorname{diam}(G) < v(C)$,

a contradiction

$\blacksquare$