Q/ If $\delta(G) \geq 2$, then how large can $g(G)$ be? n.

Q/ What if $\delta(G) \geq 3$?

<u>Thm 1</u>: $\delta(G) \geq 3 \Rightarrow g(G) < 2\log_2 n$.

<u>Dfn</u> (Moore number): For $d, g \in \mathbb{N}$, let

$$n_0(d,g) = \begin{cases} 1 + d\displaystyle\sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r+1 \text{ is odd.} \\ 2\displaystyle\sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r \text{ is even.} \end{cases}$$

<u>Obs</u>: $n_0(d,g) \geq 2\displaystyle\sum_{i=0}^{\lfloor g/2 \rfloor - 1} (d-1)^i$.

<u>Thm 2 (Moore)</u>: $\delta(G) \geq \delta$ and $g(G) \geq g \Rightarrow v(G) \geq n_0(\delta, g)$.

<u>proof that thm 2 $\Rightarrow$ thm 1</u>:

Let $g = g(G)$, $r = \lfloor \frac{g}{2} \rfloor$ and $n = v(G)$.

If $g = 3$, then we are done : $2 \cdot \log_2 n \geq 2 \log_2 3 > 3$.

Assume $r \geq 2$. Moore's thm $\oplus$ Obs $\oplus$ $\delta(G) \geq 3 \Rightarrow$

$$n \geq 2 \sum_{i=0}^{r-1} 2^i = 2 \cdot (2^r - 1) > 2^{r + \frac{1}{2}} \geq 2^{g/2}$$

$$\Rightarrow \log_2 n > g/2$$

$\blacksquare$

Proof of Moore's thm (odd girth):

$g = 2r+1$ (girth) (why is $g(G) < \infty$?)

v_0 so that $d_G(v_0, x) = \mathrm{diam}(G) \geq \frac{g-1}{2} = r$ for some x.

For $i \geq 0$, $N_i := \{v \in V(G) : d_G(v, v_0) = i\}$ (ith layer)

Obs: $N_i \cap N_j = \phi \ \forall i \neq j$.

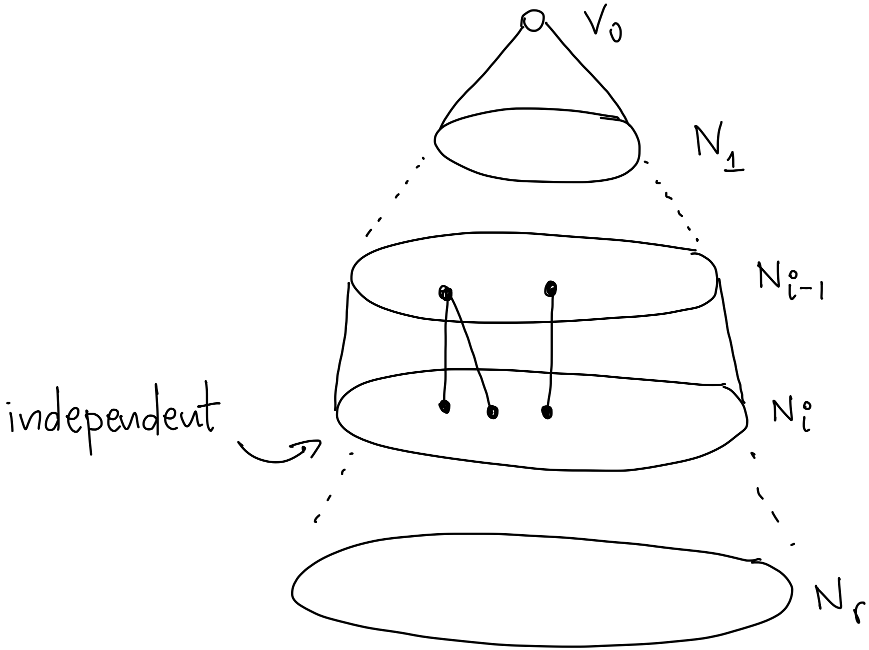
$\Rightarrow v(G) \geq |N_0| + |N_1| + \dots + |N_r|$. (*)

Notation: $N_{\leq i} := \bigcup_{j=0}^{i} N_j$

Lemma 1: If $i \leq r$ and $v \in N_i$, then

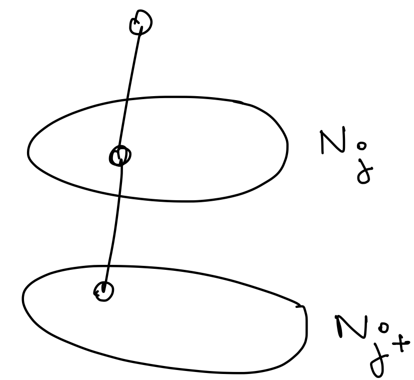
$$\left| N_G(v) \cap N_{\leq i-1} \right| = \left| N_G(v) \cap N_{i-1} \right| = 1$$

Lemma 2: N_i is an ind. set $\forall\, i \leq r-1$.



proof that L1 + L2 + $\circledast$ $\Rightarrow$ Moore's bound :

If $j \le r-1$, then



$$(\delta-1)|N_j| \leqslant e(G[N_j, N_{j+1}]) \leqslant |N_{j+1}|$$

By induction,

$$|N_1| \geqslant \delta, \quad |N_2| \geqslant (\delta-1)\delta, \quad \dots, \quad |N_r| \geqslant (\delta-1)^{r-1}\delta .$$

$$\Rightarrow \quad v(G) \geqslant 1 + \delta \sum_{j=0}^{r-1} (\delta-1)^j$$

$\blacksquare$

proof of L1 : (Recall prop. in L2)

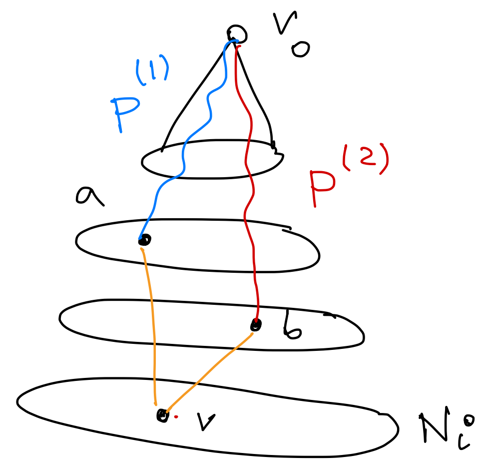
- $|N(v) \cap N_{i-1}| \geq 1$

  Let $v_0 v_1 \cdots v_i = v$ be the shortest $v_0 - v$ path. Then, $v_{i-1} \in N_{i-1}$. ✓

  $\uparrow = d(v_0, v)$.

- Suppose for contradiction that

  $\exists\, a \neq b$ st $a, b \in N_{\leq i-1} \cap N(v)$.



- $P^{(1)} :=$ shortest $v_0 - a$ path

- $P^{(2)} :=$ shortest $v_0 - a$ path.

- $P^{(1)} \cup av$ and $P^{(2)} \cup bv$ are 2 distinct $v_0 - v$ paths.

  $\Rightarrow P^{(1)} \cup av \cup P^{(2)} \cup bv \supseteq$ a cycle C

- $|C| \leq 2(i-1)+2 = 2i \leq 2r < g$. ※

Proof of L2:

Suppose $\exists$ edge $ab \in G[N_i]$

$P^{(1)} :=$ shortest $v_0 - a$ path

$P^{(2)} :=$ —— $v_0 - b$ ——

$\exists$ cycle of length $2i+1 \leq 2(r-1)+1 < g$ ※

<u>Defn</u>: G is connected if $\forall\, x, y \in G\ \exists$ x-y path in G.

<u>Defn</u>: For $x \in V(G)$,

$$C_x := \{ y \in V(G) : \exists\ x\text{-}y \text{ path in } G \}$$

is the (connected) component of x.

A set D is a (connected) component if $\exists\ x$ st $D = C_x$.

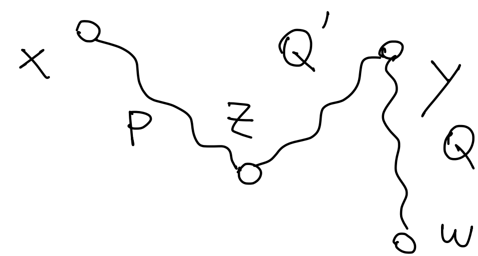
Lemma: $C_x \cap C_y \neq \emptyset \iff C_x = C_y$.

proof: $(\Rightarrow)$

Let $z \in C_x \cap C_y$ and P an x-z path.

Let $w \in C_y$ and Q an w-y path

$Q' = y$-z path.



x-w walk : $P \cup Q' \cup Q$

$\Rightarrow$ x-w path in G.

$\Rightarrow w \in C_x \Rightarrow C_y \subseteq C_x$. | Similarly, $C_x \subseteq C_y$ $\blacksquare$

Defn : A graph is acyclic if it has no cycle.

Tree : Connected acyclic graph

Leaf : vtx of degree 1 in a tree.

Claim : Every tree on at least 2 vtxs has a leaf.

proof 1 : Suppose not. Min degree $\geq 2 \Rightarrow \exists$ cycle, contradiction.

$\blacksquare$

proof 2 : Take the longest path. Its endpoints have degree 1.

$\blacksquare$