Recall: Decision problem: a BF: $f : X \to \{0,1\}$

Defn: A DP $f : X \to \{0,1\}$ is in the class **P** if $\exists$ an algorithm that for every $x \in X$ outputs $f(x)$ in time polynomial in $|X|$.

Example: $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = \begin{cases} 1, & \text{if } x \text{ is prime} \\ 0, & \text{oth.} \end{cases}$ is in **P** (Agrawal-Kayal-Saxena)

Defn: A DP $f : X \to \{0,1\}$ is in **NP** if $\exists$ a verifier $V \in \mathbf{P}$ and a polynomial p such that $\forall x \in X$:

$$f(x) = 1 \iff \exists \text{ witness } w \text{ with } |w| \leq p(|x|) \text{ and } V(x,w) = 1.$$

Example: Hamiltonicity is in NP.

Defn: A DP $f : X \to \{0,1\}$ is in co-**NP** if $-f+1 \in \mathbf{NP}$.

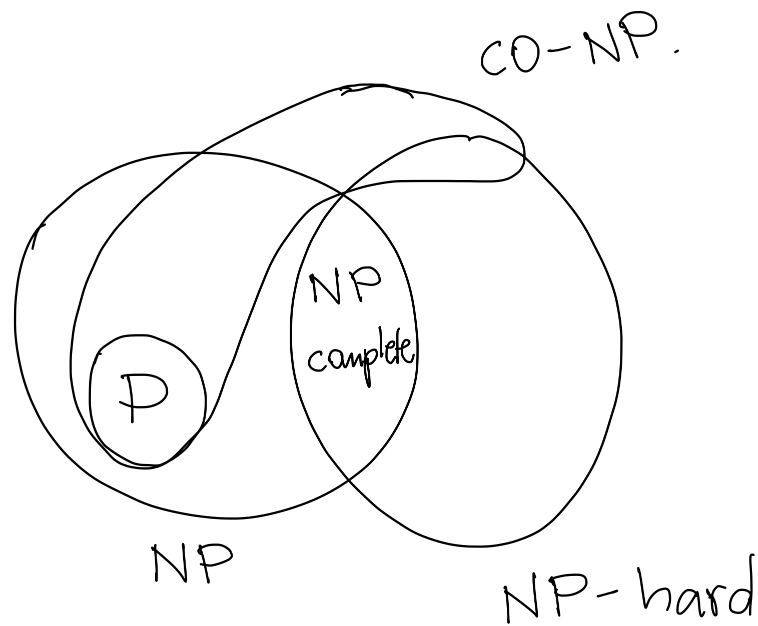
Obs: $\mathbf{P} \subset \mathbf{NP} \cap \text{co-}\mathbf{NP}$

Defn: A DP $f : X \to \{0,1\}$ is **NP**-hard if $\forall\, g : Y \to \{0,1\} \in$ **NP** $\exists$ polynomial time computable mapping $r : Y \to X$ such that $\forall\, y$

$$g(y) = 1 \iff f(r(y)) = 1 .$$

Defn: **NP**-complete = **NP**-hard $\cap$ **NP**

Not known: N = NP and NP = co-NP.


CO-NP.
NP
complete
P
NP
NP-hard

GRAPH SCANNING ALGORITHM (GSA)

Input: A graph G and $s \in V(G)$.

Set $i = 0$, $R_0 := \{s\}$, $Q_0 := \{s\}$ and $T_0 := \phi$, $\ell(s) = 0$

While $Q_i \neq \phi$, do:

choose $v \in Q_i$ and set $v_i = v$

If $R_i^c \cap N(v) \neq \phi$, then

choose $w \in R_i^c \cap N(v)$.

set $R_{i+1} := R_i \cup \{w\}$, $Q_{i+1} := Q_i \cup \{w\}$ and $T_{i+1} := T_i \cup \{vw\}$, $\ell(w) = \ell(v) + 1$.

Else, set $R_{i+1} := R_i$, $Q_{i+1} = Q_i \setminus \{v\}$ and $T_{i+1} := T_i$

$i \leftarrow i+1$.

Output: (R_i, T_i).

Theorem: The GSA outputs a spanning tree of the connected component of G containing s.

Claim 1: $Q_i \subseteq R_i \subseteq R_{i+1} \subseteq V(G)$ and $T_i \subseteq T_{i+1} \subseteq E(G)$ $\forall i$.

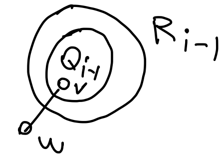
proof: induction $\oplus$ lines 4 and 5. $\blacksquare$

Claim 2: The algorithm works correctly.

proof: $2|Q_i| - |R_i| < 2|Q_{i+1}| - |R_{i+1}| \leq 2V(G)$. $\blacksquare$

Claim 3: (R_i, T_i) is a tree $\forall i$.

proof: True for $i = 0$. Suppose (R_{i-1}, T_{i-1}) is a tree and $Q_{i-1} \neq \emptyset$ for some $i \geq 1$.

If GSA runs line 4, $(R_i^\circ, T_i^\circ) =$ 

$\leftarrow$ connected acyclic

———— 5, $(R_i^\circ, T_i^\circ) = (R_{i-1}^\circ, T_{i-1}^\circ)$ ———— $\blacksquare$

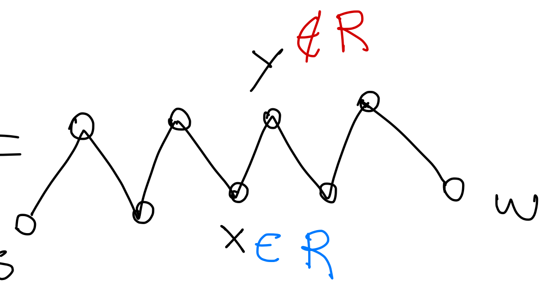
<u>proof of thm:</u> $H := (R, T) =$ output of GSA

$C_s =$ connected component of s in G.

Claims 1–3 $\Rightarrow R \subseteq C_s$.

GOAL: $C_s \subseteq R$.

Suppose for contradiction that $\exists\, w \in C_s \setminus R$.

$P =$ s–w path $=$ 

$\exists\, i$ st $R_{i+1} = R_i \cup \{x\}$. $\Rightarrow Q_{i+1} = Q_i \cup \{x\}$.

As GSA stops, $\exists\, T$ st $Q_T = \phi$.

$\Rightarrow \exists\, j$ such that $Q_{j+1} = Q_j \setminus \{x\}$.

As this can only happen in line 5, we must have

$R_j^c \cap N(x) = \phi \Rightarrow \{y\} \subseteq N(x) \subseteq R_j \subseteq R$,

a contradiction. $\blacksquare$

Running time of GSA $= O(n \cdot m)$.