GRAPH SCANNING ALGORITHM (GSA)

Input: A graph G and $s \in V(G)$.

Set $i = 0$, $R_0 := \{s\}$, $Q_0 := \{s\}$ and $T_0 := \phi$, $\ell(s) = 0$

While $Q_i \neq \phi$, do:

choose $v \in Q_i$ and set $v_i = v$

If $R_i^c \cap N(v) \neq \phi$, then

  choose $w \in R_i^c \cap N(v)$.

  set $R_{i+1} := R_i \cup \{w\}$, $Q_{i+1} := Q_i \cup \{w\}$ and $T_{i+1} := T_i \cup \{vw\}$, $\ell(w) = \ell(v) + 1$.

Else, set $R_{i+1} := R_i$, $Q_{i+1} = Q_i \setminus \{v\}$ and $T_{i+1} := T_i$

$i \leftarrow i+1$.

Output: (R_i, T_i).

Breadth First search: The GSA where the data structure of Q is a queue.

"First in - first out strategy"

Depth First search: The GSA where the data structure of Q is a stack.

"Last in - first out strategy".

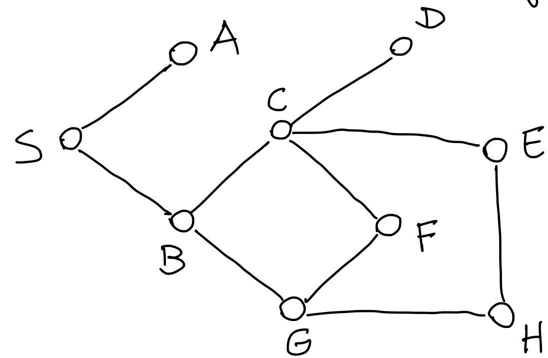
More formally, in a GSA Q_i is a vector and in ④ $Q_{i+1} = (Q_{i-1}, w)$

BFS: in ①, $v = Q_i[1]$

DFS: in ①, $v = Q_i[\ell]$, where ℓ is the length of Q_i.

Example:

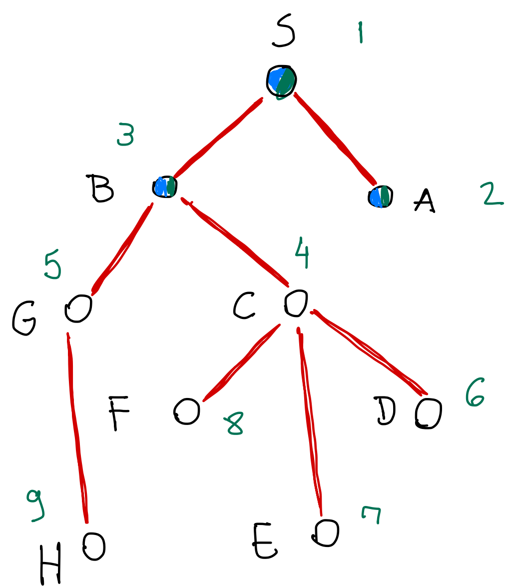
$G =$



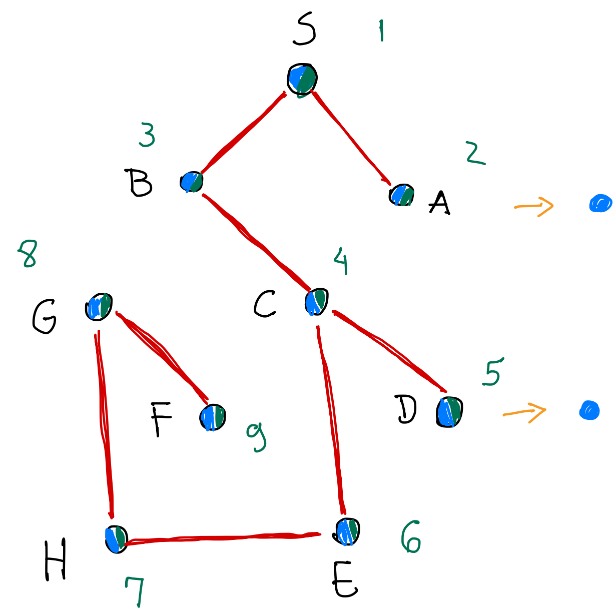
BFS for G and s

● = vtxs in Q ⟋ = edges in T

● = vtxs in R



DFS for G and s.



Lemma: Let G be a connected graph and $s \in V(G)$.
If T is the tree output by $BFS(G, s)$, then

$$d_G(s,u) = d_T(s,u) \quad \forall\, u \in V(G).$$

In particular, $|d_T(s,u) - d_T(s,v)| \leq 1 \quad \forall\, uv \in E(G)$.

Prop1: ℓ is well-defined.

proof: $\forall\, w \in V(G)$, $\exists\, i$ st $R_{i+1} = R_i \cup \{w\}$.

$$\Rightarrow w \in R_j \;\; \forall\, j \geq i+1 \;\Rightarrow w \notin R_j^c \;\; \forall\, j \geq i+1.$$

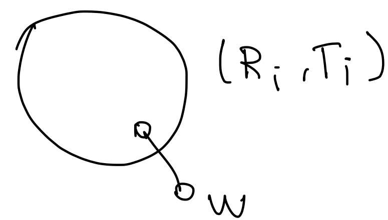
$\blacksquare$

Prop 2 : $\ell(v) = d_{(R_i,T_i)}(s,v) \ \forall v \in R_i \ \forall i$.

proof: induction in i.

Note that either $(R_{i+1}, T_{i+1}) = (R_i, T_i)$ or

$(R_{i+1}, T_{i+1}) =$



$\blacksquare$

Prop 3 : $\ell(w) \leq \ell(v_i) + 1 \ \forall w \in R_i$.

proof : Note that $\ell(v_j) \leq \ell(v_{j+1}) \ \forall j$.

$w \in R_i \Rightarrow \exists j \leq i$ st $\ell(w) = \ell(v_j) + 1 \leq \ell(v_i) + 1$

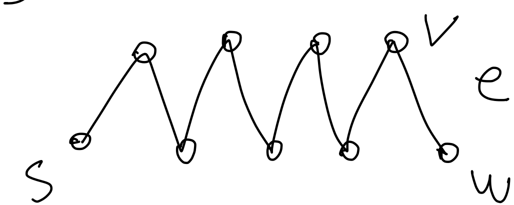
$\blacksquare$

proof of lemma: $d_G(s,u) \le d_T(s,u)$ ✓

Suppose for contradiction that

$$B = \{x : d_G(s,x) < d_T(s,x)\} \neq \emptyset$$

Take $w \in B$ st $d_G(s,w) = \min_{x \in B} d_G(s,x) =: d$



P = shortest $s-w$ path in G

$d_G(s,v) = d-1 \Rightarrow v \notin B$

$\Rightarrow d_G(s,v) = d_T(s,v). \Rightarrow e \notin T.$

$$\ell(w) = d_T(s,w) > d_T(s,w) = d_G(s,v) + 1$$
$$= \ell(v) + 1$$

Let i be such that $v_i = v$. This ⊕ Prop 3 $\Rightarrow w \notin R_i$

$\Rightarrow$ e will be added to T, a contradiction

$\boxtimes$

<u>Lemma</u>: Let G be a connected graph and $s \in V(G)$.
If T is the tree output by DFS(G, s), then
$\forall\, uv \in E(G)\ \exists$ a s-to-leaf path $P \subseteq T$ with $u, v \in V(P)$

<u>Claim</u> : Q_i is a path $\forall i$.

Suppose $i := \min\{t : v \in Q_t\} < \min\{t : u \in Q_t\}$

Let $j = \max\{t : v \in Q_t\}$.

$\Rightarrow$ $v_j = v$ and $Q_{j+1} = Q_j \setminus \{v\}$

$\Rightarrow$ $R_j^c \cap N(v) = \phi$ $\Rightarrow$ $u \subseteq N(v) \subseteq R_j$

$\Rightarrow$ $\exists$ $i \le k \le j$ such that $u \in Q_k$.

$\Rightarrow$ Q_k is a path containing u and v

Extend Q_k to an s-to-leaf path. $\blacksquare$