Minimum spanning tree (MST)

Q/ G graph and $c: E(G) \to \mathbb{R}$. Is there a polynomial time algorithm to find $T \subseteq G$ spanning tree with minimum weight $c(T) := \sum_{e \in E(T)}' c(e)$?

Kruskal's algorithm (KA)

Input: G connected, $c: E(G) \to \mathbb{R}$.

Sort the edges st $c(e_1) \leq \cdots \leq c(e_m)$

Set $i := 0$ and $T_i := (V(G), \emptyset)$.

While $i < m$, do:

If $T_i + e_{i+1} \not\supseteq$ cycle, then

Set $T_{i+1} := T_i + e_i$

Else set $T_{i+1} := T_i$

Output: T_m.

RUN DFS $O(n)$ $\Big\}$ m times

Thm : Kruskal's algorithm outputs a spanning tree of minimum weight and its running time is $O(nm)$.

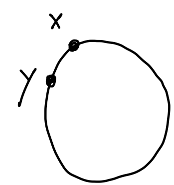
Notation : T_{xy} = the unique x-y path in a tree T.

Lemma : T is a MST in $G \iff \forall\ xy \in E(G) \setminus E(T)$ and $uv \in T_{xy}$, $c(uv) \leq c(xy)$.

proof of thm, assuming lemma :

Let T be the output of KA.

Let $xy \in E(G) \setminus E(T)$ and i st $e_{i+1} = xy$.



$P \in T_i + e_{i+1}$

As $xy \notin E(T)$, $\exists$ cycle in $T_i + e_{i+1}$

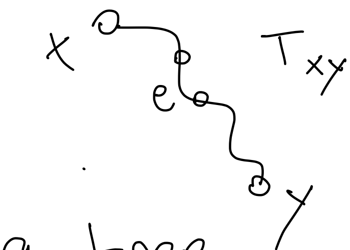
$\Rightarrow P \subseteq T_i \Rightarrow P = T_{xy}$

On the other hand, $P \subseteq T_i \Rightarrow$

$c(uv) \leq c(xy)\ \forall\ uv \in E(P) = E(T_{xy})$. $\blacksquare$

proof of Lemma:

($\Rightarrow$) Suppose for contradiction that $\exists \ xy \in E(G) \setminus E(T)$ and $e \in T_{xy}$ st $c(e) > c(xy)$.

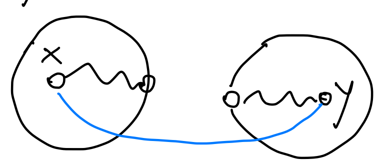


$T' := T - e + xy \Rightarrow c(T') < c(T)$.

Goal: To show that T' is a spanning tree.

Claim 1: T' is connected.

proof: T minimally connected $\Rightarrow$ $T - e$ contains 2 components.



xy connects these!

$\blacksquare$

__Claim 2__ : $e(T') = e(T) = v(G) - 1$ $\blacksquare$

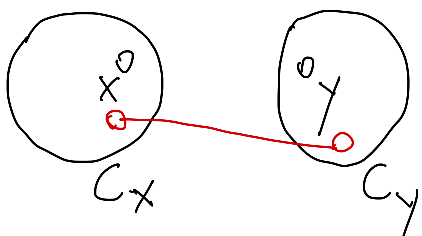
$\Rightarrow T'$ is a tree.

($\Leftarrow$) Let T' be MST of G with

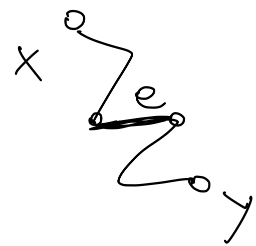
$$|E(T') \cap E(T)| = \max\{|E(T'') \cap E(T)| : T'' \text{ MST}\}$$

Suppose for contradiction that $T' \neq T$.

Let $xy \in E(T') \setminus E(T)$.



$T' - xy$



T

- $\exists\ e \in T_{xy}$ such that $|e \cap C_x| = |e \cap C_y| = 1$
- $c(e) \leq c(xy)$ by assumption.
- $T'' = T' - xy + e$ is a spanning tree with
  $c(T'') \leq c(T') \Rightarrow T''$ is MST.
- $|E(T'') \cap E(T)| > |E(T') \cap E(T)|$,

  contradiction

$\blacksquare$

Euler tours

Defn: A walk is an alternating sequence of vtxs and edges $v_1 e_1 v_2 e_2 \cdots v_k e_k v_{k+1}$ such that $e_i = v_i v_{i+1}$ $\forall i$.

$\uparrow$ length of the walk.

The walk is closed if $v_1 = v_{k+1}$. It is simple if $e_i \neq e_j$ $\forall i \neq j$.

Notation: $V(W) = \{v_1, \dots, v_{k+1}\}$, $E(W) = \{e_1, \dots, e_k\}$ and $G_W = (V(W), E(W))$

Defn: A walk W in a graph G $(E(W) \subseteq E(G))$ is an Euler tour if W is closed, simple and $G_W = G$.

Eulerian = a graph in which $\exists$ an Euler tour in it.

Thm: A connected graph G is Eulerian $\iff$ $d_G(v) \equiv 0 \pmod 2$ $\forall v \in V(G)$.

<u>Claim 1</u>: Let W be an Euler tour. Then, $d_{G_W}(v) \equiv 0 \pmod 2$ $\forall v \in V(G_W)$.

<u>Proof</u>: $d_{G_W}(v) = 2 \cdot \#\{i : v_i = v\}$ ▨

<u>Claim 2</u>: $d_G(v) \equiv 0 \pmod 2$ $\forall v \in V(G) \Rightarrow$ Every maximal simple walk in G is closed

<u>proof</u>: $W = v_1 e_1 v_2 e_2 \cdots v_k e_k v_{k+1}$ maximal and simple in G.

Suppose for contradiction that $v_1 \neq v_{k+1}$. Then, $d_{G_W}(v_1)$ is odd.

$\Rightarrow \exists\ e \in E(G) \setminus E(G_W)$ st $v_1 \in e$

$\Rightarrow W$ is not maximal, a contradiction.

▨

<u>proof of thm</u>: Claim 1 gives $(\Rightarrow)$.

$(\Leftarrow)$ Let W be a maximum simple walk in G. Claim 2 $\Rightarrow$ W is closed.

Suppose for contradiction that $G_W \neq G$.

G connected $+$ $G_W \neq G$ $\Rightarrow$ $\exists\ uv \in E(G)\setminus E(G_W)$ with $u \in V(G_W)$. $\}\ (*)$

$d_{G-G_W}(x) \equiv 0 \pmod{2}\ \forall x \in V(G)$ $\}\ (**)$.

$(*) + (**)$ $\Rightarrow$ $\exists$ maximal simple (and closed) walk W' in $G - G_W$ starting at u, with $E(G_{W'}) \neq \emptyset$.

Take a walk W'' in G_W starting at u st $G_{W''} = G_W$.

Then $W'W''$ is a longer walk, contradiction. $\blacksquare$