Hamiltonian cycles (H.C.)

Thm (Dirac '52) $\delta(G) \geq v(G)/2 > 1 \Rightarrow G$ is Hamiltonian.

$\checkmark$

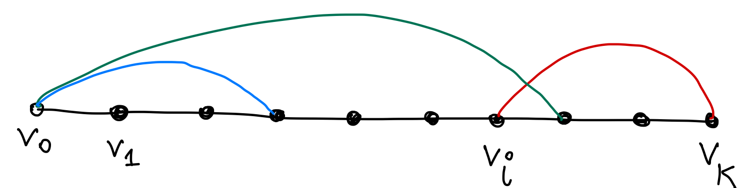
proof:

Claim: G is connected.

proof: If not, $\exists$ connected component C with $v(C) \leq n/2$

$\Rightarrow \exists\, v \in V(C)$ st $d_G(v) \leq n/2 - 1$, contradiction. $\blacksquare$

P = maximum path in G = $v_0 v_1 \cdots v_K$



$N_K = \{ i : v_i \in N(v_K) \} \subseteq \{0, 1, \ldots, K-1\}$

$N_0 = \{ i : v_i \in N(v_0) \}$ and $N_0 - 1 = \{ i-1 : i \in N_0 \} \subseteq \{0, 1, \ldots, K-1\}$

Obs: $N_K \cap (N_0 - 1) \neq \phi \Rightarrow \exists\, (K+1)$-cycle in $G[V(P)]$.

$$|N_K \cap (N_0 - 1)| = |N_K| + |N_0 - 1| - |N_K \cup (N_0 - 1)| \geq n - K \geq 1$$

$K \leq n-1$

G is connected ⊕ P is maximum ⊕ $G[V(P)]$ is Hamiltonian $\Longrightarrow$ $V(P) = V(G)$. Then, G is Hamiltonian.

$\blacksquare$

<u>Defn</u>: For a graph on n vtxs and vxts with degrees $d_1 \leq \cdots \leq d_n$, we call $s_G := (d_1, \cdots, d_n)$ the <u>degree sequence</u> of G.

<u>Notation</u>: $(x_1, \cdots, x_n) \leq (y_1, \cdots, y_n)$ if $x_i \leq y_i \; \forall i$.

<u>Defn</u>: $(a_1, \cdots, a_n)$ is <u>Hamiltonian</u> if

$$s_G \geq (a_1, \cdots, a_n) \Rightarrow G \text{ is Hamiltonian.}$$

<u>Obs</u>: $(\underbrace{n/2, \cdots, n/2}_{n \text{ times}})$ is Hamiltonian.

Thm (Chvátal '72) Let $n \geq 3$ and $0 \leq a_1 \leq \cdots \leq a_n < n$, $a_i \in \mathbb{Z}\ \forall i$.

$(a_1, \dots, a_n)$ is Hamiltonian $\iff$ $\underbrace{\forall i < n/2 \text{ st } a_i \leq i, \text{ we have } a_{n-i} \geq n-i.}_{(*)}$

proof:

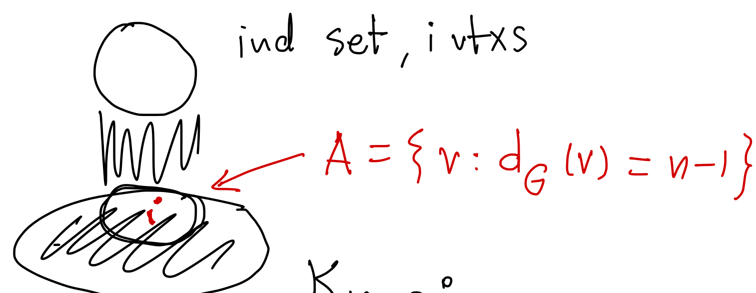
$(\Rightarrow)$ Suppose for contradiction that $\exists\, i < n/2$ st $a_i \leq i$ and $a_{n-i} \leq n-i-1$.

Goal: to find a non-Hamiltonian G with $s_G \geq r \geq (a_1, \dots, a_n)$.

Obs: $r := (\underbrace{i, \dots, i}_{i \text{ times}}, n-i-1, \dots, n-i-1, \underbrace{n-1, \dots, n-1}_{i \text{ times}}) \geq (a_1, \dots, a_n)$.

G : $s_G = r$

$=$



$G - A$ has $i+1$ connected components.

<u>Claim</u>: C Hamiltonian cycle and $I \subseteq V(C) \Rightarrow$

$C - I$ has at most $|I|$ components.

<u>proof</u>: induction on $|I|$ ▨

$(\Leftarrow)$ Suppose not. $\mathcal{B} := \{ G \text{ non-Hamiltonian} : s_G \geq (a_1, \dots, a_n) \} \neq \emptyset$

Take $G \in \mathcal{B}$ so that $e(G) = \max \{ e(H) : H \in \mathcal{B} \}$.

<u>Claim</u>: H st $s_H = (d_1, \dots, d_n) \geq (a_1, \dots, a_n) \Rightarrow s_H$ satisfies $(*)$

<u>proof</u>: Suppose that $i < n/2$ st $d_i \leq i$. But $a_i \leq d_i \leq i \Rightarrow d_{n-i} \geq a_{n-i} \geq n-i$ ▨

$\mathcal{P} = \{ (u,v) : uv \notin E(G) \text{ and } d(u) \leq d(v) \}$

Take $(x,y) \in \mathcal{P}$ st $d(x) + d(y) = \max \{ d(u) + d(v) : (u,v) \in \mathcal{P} \}$

For simplicity, denote $r := d(x)$.

<u>Lemma 1</u>: $d(x)+d(y) \leqslant n-1$, and hence $r < n/2$.

<u>Lemma 2</u>: $\#\{v : d(v) \leqslant r\} \geqslant r$.

Lemma 1 + Lemma 2 $\Rightarrow$ Thm($\Leftarrow$): Lemmas $\Rightarrow d_r \leqslant r < n/2$.

As S_G satisfies $\circledast$, we have

$$n-r \leqslant \underbrace{d_{n-r} \leqslant d_{n-r+1} \leqslant \cdots \leqslant d_n}_{\text{size } r+1}$$

$d(x)=r \Rightarrow \exists\ z \notin N(x)$ such that $d(z) \geqslant n-r$.

$$\Rightarrow d(x)+d(z) \geqslant n > d(x)+d(y),$$

contradiction.

Proof of L1 :

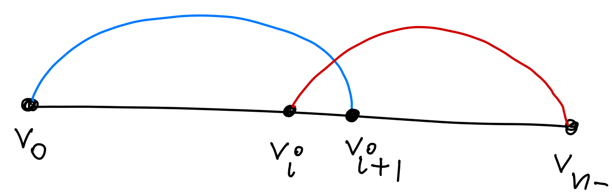
$S_{G+xy} \geq S_G \geq \vec{a}$. Then, $G+xy \supseteq$ H.C.

$\Rightarrow \exists$ path $P = v_0 v_1 \cdots v_{n-1} \subseteq G$ with $v_0 = x$ and $v_{n-1} = y$.

$N_0 = \{ i : v_i \in N(v_0) \}$ and $N_{n-1} = \{ i : v_i \in N(v_{n-1}) \}$

Obs : $(N_0 - 1) \cap N_{n-1} = \emptyset$

(otherwise H.C. $\subseteq G$)



$N_0 - 1,\ N_{n-1} \subseteq \{0, \dots, n-2\}$

$\Rightarrow \quad d(x) + d(y) = |N_0 - 1| + |N_{n-1}| = |(N_0 - 1) \cup N_{n-1}| \leq n-1$

$\Rightarrow \quad d_G(v_0) < n/2$

$\blacksquare$

Proof of L2: We claim that $d(v_i) \leq d(x)$ $\forall\ i \in N_0 - 1$.

$i \in N_0 - 1 \Rightarrow i \notin N_{n-1} \Rightarrow v_i y \notin E(G)$.

$\Rightarrow d(x) + d(y) \geq d(v_i) + d(y)$

$\blacksquare$