Travelling salesperson problem (TSP)

Given $c : E(K_n) \to \mathbb{R}_{>0}$, find a H.C. $C \subseteq K_n$ whose weight $c(C) := \sum_{e \in E(C)} c(e)$ is minimum.

Metric TSP

Given $c : E(K_n) \to \mathbb{R}_{>0}$ such that $c(xy) + c(yz) \geq c(xz)$ $\forall x, y, z \in V(K_n)$, find

$OPT(K_n, c) :=$ H.C. $C \subseteq K_n$ of minimum weight.

stands for OPTIMAL. Common notation for many problems in CS.

Defn: A is a *ρ-approximation algorithm* for $f : I \to \mathbb{R}_{>0}$ if $A(x) \leq \rho \cdot f(x)$ $\forall x$.

Notation: $MST(K_n, c) :=$ spanning tree $T \subseteq K_n$ whose weight $c(T)$ is minimum.

Lemma: $c(MST) \leq c(OPT)$

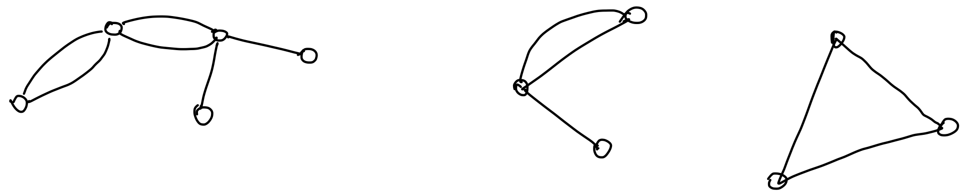
proof: $C = OPT(K_n, c)$ and $e \in E(C)$.

$C - e$ is a tree $\Rightarrow c(MST) \leqslant c(C-e) \leqslant c(OPT)$

$\blacksquare$

Multigraph: a pair (V, E) where E is a multiset such that $e \subseteq X$ and $|e| = 2$ $\forall e \in E$.

Examples:



The double tree algorithm (DTA)

Input: K_n and $c : K_n \to \mathbb{R}_{\geqslant 0}$.

① Find a MST T

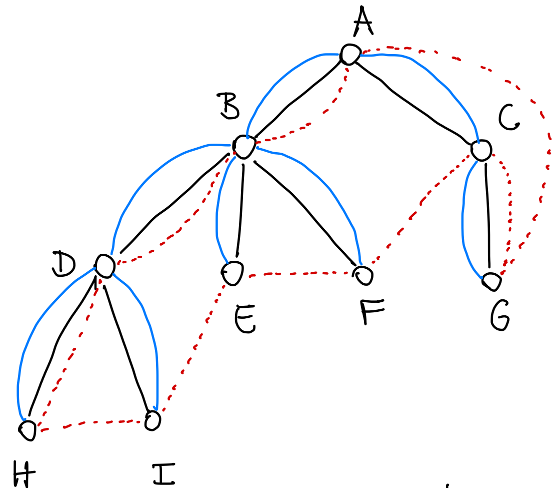
② Duplicate T

③ Take an Eulerian tour W in T (duplicated)

④ Shortcut W to obtain a H.C. C.

Output: C.

Example :



Euler tour (w/o edges) : A B D H D I D B E B F B A C G C A | H.C : A B D H I E F C G A

Lemma : $C = DTA(K_n, c) \Longrightarrow c(C) \leq 2\,c(MST)$.

Corollary : DTA is a 2-approximation algorithm for MTS.

Christofides' Algorithm

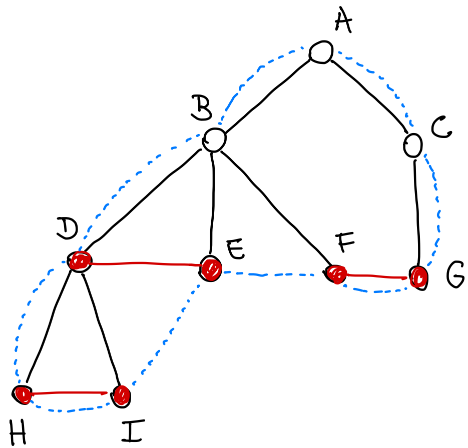
Input: K_n and $c : E(K_n) \to \mathbb{R}_{>0}$ metric.

Running time $= O(n^3)$
(no proof here)

1. Find MST T w.r.t. (K_n, c).
2. $\sigma := \{v : d_T(v) \equiv 1 \pmod 2\}$
3. Find a minimum weight perfect matching $M \subseteq K_n[\sigma]$ w.r.t. c.
4. $G := (V(K_n), E(T) \cup E(M))$ and W be Eulerian tour in G.
5. Shortcut W to obtain a H.C. C

Output: C

Example:



G

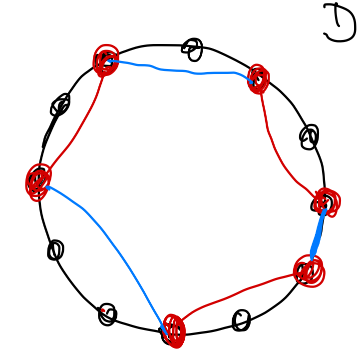
Euler tour (without edges): ABDHIDEBFGCA

Output: ABDHIEFGCA

Note that $c(C) \leq c(T) + c(M) = c(MST) + c(M)$.

Claim: $c(M) \leq c(OPT)/2$.

<u>proof</u> : $D = OPT$ = H.C. of minimum weight.



D' = an arbitrary H.C. in $K_n[O]$.

$c : E(K_n) \to \mathbb{R}$ metric $\Rightarrow$ $c(D') \leq c(D)$.

Partition $E(D')$ into two matchings, say M_1 and M_2.

$V(D') =$

$M_1 =$

$M_2 =$

Then, $c(M) \leq \dfrac{c(M_1) + c(M_2)}{2} \leq \dfrac{c(D)}{2}$.

We just proved :

<u>Thm</u> : Christofides' Algorithm is a 3/2-approximation algorithm for TSP.