Lecture Notes: Discrete Structures I

Winter Semester 2025/26

Lecture by Leticia Mattos

Script created by Lucian von Hagen

Foreword

This script is not official and is based on my personal notes from the lecture. I cannot guarantee that it is complete or free of errors.

Comments, suggestions for corrections or remarks to make the script better are highly appreciated. Please send them to me via email: `lucian.hagen@stud.uni-heidelberg.de`

1 Graphs

Definition 1.1 **Graph**

A graph is a pair of sets (V, E) (V - vertex set, E - edge set) so that $E \subseteq \{\{x, y\} : x, y \in V \wedge X \neq Y\}$ and $|V| < \infty$

For an edge $\{x, y\}$ we also just write xy.

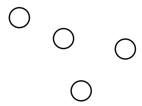
we don't allow:
- loops
- double edges
- there are no directed edges

Definition 1.2

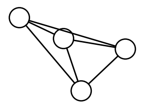
Let G be a graph ($G = (V, E)$).
- For $e = xy \in E$, x and y are the **endpoints** of e
- $x, y \in V(G)$ are **adjacent** if $xy \in E(G)$
- $e \in E(G)$ is **incident with** $v \in V(G)$ if $v \in e$
- y is a **neighbor** of x if $xy \in E(G)$; Notation $x \sim y$
- $N_G(x) = \{y : xy \in E(G)\}$ the **neighborhood** of x
- $d_G(x) = |N_G(x)|$ the **degree** of x
- $v(G) = |V(G)|$ is the **order** of G and $e(G) = |E(G)|$ is the **size** of G

Definition 1.3 **Special types of graphs**

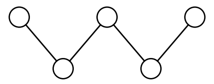
- Empty graph of n vertices (no edges)



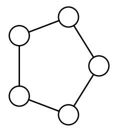
- K_n: Complete graph of n vertices (all possible edges)
  This has $\#$ edges $= \frac{n(n-1)}{2} = \binom{n}{2}$



- P_n: Path of **length** ($\#$ edges) n (no intersections)



- C_n: Cycle of **length** n (only first and last vertex are equal)



Definition 1.4 **Graph isomorphism**

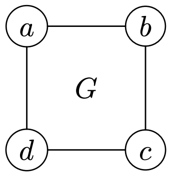
Let H, G be graphs. We say that $f : V(G) \to V(H)$ is an **isomorphism** if f is a bijection and

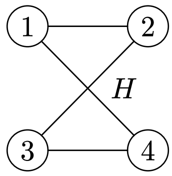
$$uv \in E(G) \iff f(u)f(v) \in E(H)$$

H and G are **isomorphic** if $\exists f \in H^G : f$ isomorphic

Notation: $H \cong G$.

When $H = G$, we say that f is an **automorphism**.



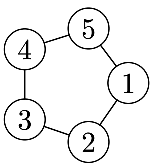


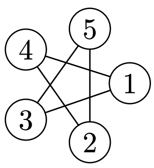
are isomorphic via $f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$.

Abuse of notation: "$P_n \subseteq G$" means $\exists H \subseteq G : H \cong P_n$

Definition 1.5 **Key definitions**

Let H, G be graphs.
1. H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
   Notation: $H \subseteq G$.
2. For $U \subseteq V(G)$, $G[U] := (U, \{xy : xy \in E(G), x, y \in U\})$ is the **subgraph** of G **induced by** U.
3. $H \subseteq G$ is an **induced subgraph of** G if $H = G[U]$ for some $U \subseteq V(G)$.
4. **Complement** of G: $\overline{G} := (V(G), E')$ with $E' = \{xy : xy \notin E(G); x, y \in V(G); x \neq y\}$





G $\overline{G}$

Definition 1.6 **Useful notations**

Let G be a graph.

1. For $U \subseteq V(G)$, $G - U := G[V(G) \setminus U]$

   If $U = \{u\}$, we write $G - u := G - \{u\}$
2. For $F \subseteq E(G)$, $G - F := (V(G), E(G) \setminus F)$

   $\rightarrow$ This creates ambiguity between $G - \{a, b\}$ being G without vertices a, b or G without the edge ab. So it's better to not write edges in curly braces anymore.
3. $\Delta(G) := \max_{v \in V(G)} d_G(v)$ the **maximum degree**.
4. $\delta(G) := \min_{v \in V(G)} d_G(v)$ the **minimum degree**.
5. $\overline{d}(G) := \frac{1}{v(G)} \sum_{v \in V(G)} d_G(v)$ the **average degree**.

Lemma 1.7 **Handshaking Lemma**

$$\sum_{v \in V(G)} d_G(v) = 2e(G)$$

Proof: $\sum_{v \in V(G)} d_G(v) = \#\{(e, v) : e \in E(G), v \in e\} = 2e(G)$ □

Definition 1.8

H is d-regular if $d_H(v) = d \quad \forall v \in V(H)$

Q/ Given G, can you find a d-regular graph H so that $G \subseteq H$? Yes! Take $K_v(G)$!

Lemma 1.9

$\forall$ graph G $\exists H : H$ is $\Delta(G)$-regular $\wedge$ $G \subseteq H$

Proof: $G_0 := G$ and $G_0' \cong G$ such that $V(G_0) \cap V(G_0') = \emptyset$.

For $i \geq 1$:

$\forall x \in V(G_{i-1}) \wedge d_{G_{i-1}} < \Delta(G)$, connect x to its twin in G_{i-1}'.

$G_i := G_{i-1} \cup G_{i-1}'$

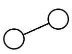
Stop if G_i is $\Delta(G)$-regular

□

Lemma 1.10

$e(G) \geq 1 \implies \exists H \subseteq G : \delta(H) > \frac{e(H)}{v(H)} \geq \frac{e(G)}{v(G)}$

Proof: per Induction over $\#$ vertices.

Base case: $G = (\{1, 2\}, \{12\})$ 

$\implies H = G$

We can assume that $\delta(G) \le \frac{e(G)}{V(G)}$ (Take $H = G$)

Claim: If $d_G(x) \le \frac{e(G)}{n}$, then $\frac{e(G-x)}{n-1} \ge \frac{e(G)}{n}$ (with $n = v(G)$)

Suppose claim holds. Apply the induction hypothesis to $G - x : \exists H \subseteq G - x : \delta(H) > \frac{e(H)}{v(H)} \ge \frac{e(G-x)}{v(G-x)} \overset{\text{claim}}{\ge} \frac{e(G)}{n}$. □

Proof (Claim): $e(G-x) = e(G) - d_G(x) \ge e(G) \cdot \left(1 - \frac{1}{n}\right)) = e(G) \cdot \frac{n-1}{n}$ □

Lecture 02 (2025/10/14)

Lemma 1.11

Let G be a graph so that $v(G) \ge 1$, then $P_{\delta(G)} \subseteq G$ and $C_l \subseteq G$ for some $l \ge \delta(G) + 1$

Proof: Take the longest path $v_0, v_1, ..., v_j, v_{j+1}, ..., v_{k-1}, v_k$

$N_G(v_0) \subseteq \{v_1, ..., v_k\}$

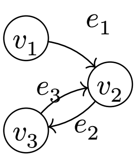
$\delta(G) \le d(v_0) = |N(v_0)| \le k = \text{size of longest path}$

$j := \max\{i : v_0 \sim v_i\}$

$N(v_0) \subseteq \left\{v_1, ..., v_j\right\} \Rightarrow \delta(G) \le j$ and $C_{j+1} \subseteq G$ □

Definition 1.12

A **walk** is an alternating sequence of vertices and edges $v_1 e_1 v_2 e_2 ... v_k e_k v_{k+1}$ so that $e_i = v_i v_{i+1}$



x-y walk is a walk with endpoints x and y.
x-y path is a path with endpoints x and y.
xPy The subpath of P starting at x and ending at y.

Proposition 1.13

Let G be a graph and $x, y \in V(G)$.

Then, $\exists$ x-y walk $\Longleftrightarrow \exists x$-$y$ path.

Proof: $\Longleftarrow$: trivial

$\Longrightarrow$: If there is a path, there is a shortest path. The shortest x-y walk can be viewed as a path. □

Definition 1.14

Distance $d_G(x, y) := \min\{l : \text{path of length } \ l \ \text{ exists between x and y}\}$
If there is no such path, $d_G(x, y) := \infty$.
Diameter $\operatorname{diam}(G) := \max_{x,y \in V(G)} d_G(x, y)$

Proposition 1.15

Let G be a graph and $G \supseteq P = x_0 ... x_k$ path.

If $d_G(x_0, x_k) = k$, then $d_G(x_0, x_i) = i = d_G(x_k, x_{k-i}) \quad \forall i \in \{0, ..., k\}$.

Proof: $d_G(x_0, x_i) \leq i$ (take $x_0 P x_i$)

Let Q be a x_0-x_i path (Goal: show that $e(Q) \geq i$)

$Q \cup x_i P x_k$ can be seen as a x_0-x_k walk

As per previous proposition, $\exists$ path $P' \subseteq Q \cup x_i P x_k$ between x_0 and x_k

$e(Q \cup x_i P x_k) = e(Q) + k - i \geq e(P') \geq k \Longrightarrow e(Q) \geq i$ □

Proposition 1.16

If $\exists$ distinct paths $P, Q \subseteq G$ with the same endpoints, then G contains a cycle.

Proof: Let x, y be the common endpoints of P and Q.

Let $P : x = v_0 v_1 ... v_k = y$
and $Q : x = u_0 u_1 ... u_l = y$

Let $i := \min\{t : v_t \notin V(Q)\}$ and $j := \min\{t > i : v_t \in V(Q)\}$

Let j' be so that $v_j = u_{j'}$

$\Longrightarrow v_{i-1} P v_j \cup u_{i-1} Q u_{j'}$ is a cycle □

Definition 1.17

Girth $g(G) := \min\{l \geq 3 : C_l \subseteq G\}$ (if no such l exists, $g(G) := \infty$)

Lemma 1.18

If G contains a cycle, then $g(G) \leq 2 \operatorname{diam}(G) + 1$.

Proof: Suppose not. $\Longrightarrow g(G) \geq 2 \operatorname{diam}(G) + 2$

Let C be a shortest cycle in G.

Then there exists $x, y \in V(C)$ so that $d_C(x, y) \geq \operatorname{diam}(G) + 1$

There exists a x-y path $P \subseteq G$ with $P \nsubseteq C$ and $e(P) \leq \operatorname{diam}(G)$

Let $P : x = v_0 v_1 ... v_k = y$ be such a path.

Let $i + 1 := \max\{t : \{v_0, ..., v_t\} \subseteq V(C)\}, j := \min\{t > i : v_t \in V(C)\}$.

Let Q be the shortest v_i-v_j path in C.

Let $C' = Q \cup v_i P v_j$.

$$\begin{aligned} e(C') &= e(Q) + e(v_i P v_j) \\ &\leq \left\lfloor \frac{v(C)}{2} \right\rfloor + \operatorname{diam}(G) \end{aligned}$$

We know, $V(C) \geq 2 \operatorname{diam}(G) + 2$

Therefore $\left\lfloor \frac{v(C)}{2} \right\rfloor + \operatorname{diam}(G) \leq \left\lfloor \frac{v(C)}{2} \right\rfloor + \frac{v(C)}{2} - 1 < v(C)$ $\square$

Lecture 03 (2025/10/20)

Q/ If $\delta(G) \geq 2$, then how large can $g(G)$ be? $\exists$ cycle of length $\leq v(G)$

Q/ What if $\delta(G) \geq 3$?

Corollary 1.19

$\delta(G) \geq 3 \Longrightarrow g(G) < 2 \log_2 v(G)$

Definition 1.20 **Moore number**

For $d, g \in \mathbb{N}$, let

$$n_0(d,g) := \begin{cases} 1 + d \sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r+1 \text{ (odd)} \\ 2 \sum_{i=0}^{r-1} (d-1)^i & \text{if } g = 2r \text{ (even)} \end{cases}$$

Observation: $n_0(d,g) \geq 2 \sum_{i=0}^{\lfloor \frac{g}{2} \rfloor - 1} (d-1)^i$ if $d \geq 2$

Theorem 1.21 **Moore**

$\delta(G) \geq \delta$ and $g(G) \geq g \Longrightarrow v(G) \geq n_0(\delta, g)$

Proof of Corollary 1.19, assuming Moore's theorem:
Let $g = g(G), r = \lfloor \frac{g}{2} \rfloor$ and $n = v(G)$

If $g = 3$, then we're done: $3 < \log_2(9) = 2\log_2(3) \leq 2 \log_2 v(G)$.

We can assume $g \geq 4$.

By Moore's theorem: $n \geq 2 \sum_{i=0}^{r-1} (\delta - 1)^i \geq 2 \sum_{i=0}^{r-1} 2^i = 2(2^r - 1) = 2^{r+1} - 2 \overset{g \geq 4}{>} 2^{r+\frac{1}{2}} = 2^{\lfloor \frac{g}{2} \rfloor + \frac{1}{2}} \geq 2^{\frac{g}{2}}$. $\square$

Proof of Moore's Theorem (odd girth): Assume G ist connected. (otherwise, apply the argument to the component with the smallest girth)

$g(G) = g = 2r + 1$

Let v_0 so that $d_G(v_0, x) = \operatorname{diam}(G) \geq \frac{g-1}{2} = r$ for some $x \in V(G)$.

Let $N_0 := \{v_0\}$, $N_1 := N_G(v_0)$, and $N_i := \{v \in V(G) : d_G(v_0, v) = i\}$ for $i \in \{2, ..., r\}$.

- Obs 1: $N_i \cap N_j = \emptyset \ \forall i \neq j$
- Obs 2: $v(G) \geq |N_0| + |N_1| + ... + |N_r|$
- Notation: $N_{\leq i} := \bigcup_{j=0}^{i} N_j$

Lemma 1.22

If $i \leq r$ and $v \in N_i$, then $|N_G(v) \cap N_{\leq i-1}| = |N_G(v) \cap N_{i-1}| = 1$

Lemma 1.23

N_i is an independent set ($I \subseteq V(G)$ with $e(G[I]) = 0$) for $i \leq r - 1$

Lemma 1.22 and Lemma 1.23 imply Moore's bond:

If $j \leq r - 1$, then

$(\delta - 1)\ |N_j| \leq e\big(G\left[N_j \cup N_{j+1}\right]\big) \leq |N_{j+1}|$

$|N_0| = 1$, $|N_1| = |N_G(v_0)| \geq \delta$, $|N_2| \geq (\delta - 1)\delta$, ..., $|N_r| \geq (\delta - 1)^{r-1}\delta$

$v(G) \geq |N_0| + ... + |N_r| \geq 1 + \delta + (\delta - 1)\delta + ... + (\delta - 1)^{r-1}\delta$ □

Proof of Lemma 1.22:
$|N_G(v) \cap N_{i-1}| \geq 1$ (follows from Proposition 1.15)

It remains to show $|N_{\leq i-1} \cap N_G(v)| \leq 1$.

Suppose this is not true. Let $v_1, v_2 \in N_G(v) \cap N_{\leq i-1}$. Let $P^{(1)}$ be a path from v_0 to v_1 and $P^{(2)}$ be a path from v_0 to v_2. Let e_1 be the edge vv_1 and e_2 be the edge vv_2.

By Proposition 1.16, $\exists$ cycle in $P^{(1)} \cup P^{(2)} \cup \{e_1, e_2\}$.

$|\text{cycle}| \leq 2(i - 1) + 2 = 2i \leq 2r < 2r + 1 = g$ □

Proof of Lemma 1.23:
Suppose not. Let $ab \in G[N_i]$ (with $a, b \in V(N_i)$).

Let $P^{(a)}$ be a path from v_0 to a and $P^{(b)}$ be a path from v_0 to b.

This means there exists a cycle in $P^{(a)} \cup P^{(b)} \cup \{ab\}$. with $|\text{cycle}| \leq 2i + 1 \leq 2(r - 1) + 1 = 2r - 1 < g$, which is a contradiction. □

2 Connectivity

Definition 2.1

A graph G is **connected** if $\forall x, y \in V(G) : \exists x\text{-}y \text{ path in } G$.

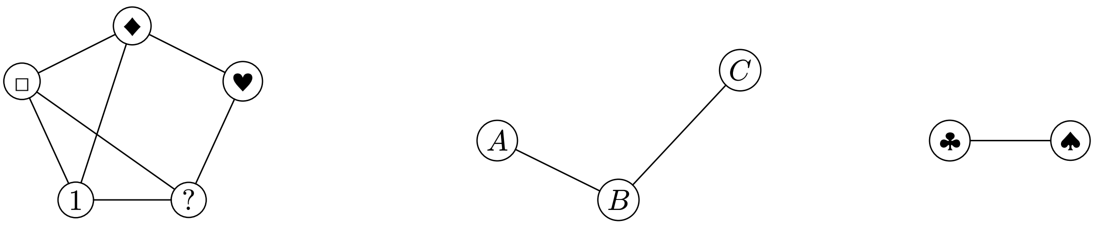
Definition 2.2

For $x \in V(G), C_x \coloneqq \{y \in V(G) : \exists x\text{-}y \text{ path in } G\}$ is the **(connected) component** of x.

Definition 2.3

A set D is a **(connected) component** of G if $\exists x \in V(G) : C_x = D$.

Example



Components: $\{1, ?, \heartsuit, \diamondsuit, \square\}, \{A, B, C\}, \{\clubsuit, \spadesuit\}$

Lemma 2.4

$C_x \cap C_y \neq \emptyset \iff C_x = C_y$

Proof of Lemma 2.4:
$\Longleftarrow$: trivial

$\Longrightarrow$: Let $z \in C_x \cap C_y$.

Let $w \in C_x$. Then, $\exists x\text{-}z \text{ path in } G$ and $\exists z\text{-}w \text{ path in } G$.

By concatenation, $\exists x\text{-}w \text{ path in } G$.

Therefore, $w \in C_y$ and $C_x \subseteq C_y$.

By symmetry, $C_y \subseteq C_x$. □

Definition 2.5

A graph is **acyclic** if it contains no cycles.

A **tree** is a connected acyclic graph.

A **leaf** is a vertex of degree 1 in a tree.

Claim

Every tree has a leaf.

Proof:
- If not, then $\delta(G) \geq 2$ and therefore $\exists$ cycle (see Lemma 1.11)

- As the longest path has two endpoints, there are at least two leafs.

□

Lecture 04 (2025/10/21)

Lemma 2.6

The following are equivalent:
(a) G is a tree
(b) G is minimally connected (G is connected but $\forall xy \in E(G) : G - xy$ is not connected)
(c) G is maximally asyclic (G is asyclic but $\forall x, y \in V(G); xy \notin E(G), x \neq y : G \cup xy$ has a cycle)
(d) $\forall u, v \in V(G), \exists! u\text{-}v$ path in G

Proof:
$(a) \implies (b)$:
Suppose (a) and $\neg(b)$.

$\exists xy \in E(G) : G - xy$ is connected.

Then there is a path P between x and y in $G - xy$. Then $P \cup xy$ is a cycle in G, which is a contradiction.

$(b) \implies (c)$:
Suppose (b) and there exists a cycle $C \subseteq G$.

Let $u, v \in V(G)$ consecutive. $G - uv$ is therefore disconnected. There are at least 2 components. This is a contradiction, as $G - uv$ must have a u-v path.

Conclusion: G minimally connected $\Rightarrow$ G is asyclic

Now, take $xy \notin E(G)$. Then $\exists x\text{-}y$ path P in G (as G is connected).

$\implies P \cup xy$ is a cycle

$(c) \implies (d)$:

Claim: $\exists \geq 1 x\text{-}y$ path in G
Proof: If $xy \in E(G)$, done.
If $xy \notin E(G)$, then $G \cup xy$ has a cycle. $\Rightarrow$ G must contain a x-y path.

Claim: $\nexists \geq 2 x\text{-}y$ paths in G
Otherwise, we would have a cycle (see Proposition 1.16)

$(d) \implies (a)$:
G is connected and asyclic, because otherwise $\exists x, y \in V(G) : \exists 2 x\text{-}y$ paths in G □

Definition 2.7

A tree $T \subseteq G$ is spanning if $V(T) = V(G)$

Lemma 2.8

G connected $\Longrightarrow \exists$ spanning tree $T \subseteq G$

Proof: $\exists T \subseteq G$ with $V(T) = V(G)$ that is minimally connected. We can achieve this by checking for every edge if removing it would disconnect the graph and removing it if not. The result is minimally connected by definition.

By Lemma 2.6 T is a tree. □

Lemma 2.9

Let T be a connected graph.

T is tree $\Longleftrightarrow e(T) = v(T) - 1$

Proof: $(\Longrightarrow)$ by induction on $v(T)$:

Base case: $v(T) = 1$. Then $e(T) = 0 = 1 - 1$.

Induction hypothesis: Let H be a tree with $v(H) = n$. Then $e(H) = n - 1$.

Induction step: Let T be a tree with $v(T) = n + 1$.
T has a leaf v.
$T - v$ is a tree on n vertices.
$\Longrightarrow e(T) = e(T - v) + 1 = (n - 1) + 1 = n$

$(\Longleftarrow)$:
Let T be connected with $e(T) = v(T) - 1$.
Let $T' \subseteq T$ be a spanning tree.
By "$\Longrightarrow$", $e(T') = v(T) - 1$

$\Rightarrow T = T'$ (because $T' \subseteq T$ and $e(T) = e(T')$ and $v(T) = v(T')$) □

3 Algorithms

Definition 3.1

An algorithm consists of a set of valid inputs and instructions so that for each valid input the computation of the algorithm is a uniquely defined finite series of **elementary steps** which produces a certain output.

Core elementary steps:
- Arithmetic operations
- Comparisons
- Variable updates
- Array access
- Evaluating one iteration of a loop
- Evaluating an if condition and taking a branch.

Input: Usually a 0-1 matrix, or a binary string.

Size of an input: $\#$ coordinates, size of the string.

Definition 3.2

Let A be an algorithm which accepts input from a finite set X.

Let $E : X \to \mathbb{N}$ be defined as $E(X) = \#$ elementary steps to compute $A(X)$

The **running time** (or time complexity) is the function $T : \mathbb{N} \to \mathbb{N}$ given by $T(n) = \max\{E(x) : x \in X, |x| = n\}$

Algorithm 3.3 **SUM**

Input: $v \in \{0, 1\}^n$ for some $n \in \mathbb{N}$
Output: integer s

$s \leftarrow 0$
for $i \in \{1, ..., n\}$, do
    $s \leftarrow s + v[i]$

return s

The number of elementary operations is surely above n and below $100n$.

Definition 3.4 **Function growth**

Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$

(1) $f = O(g)$ if $\exists C > 0 : f(n) < C * g(n)$ for all sufficiently large n
(2) $f = \Omega(g)$ if $\exists C > 0 : f(n) > C * g(n)$ for all sufficiently large n
(3) $f = \Theta(g)$ if $f = \Omega(g)$ and $f = O(g)$
(4) $f = o(g)$ if $\frac{f(n)}{g(n)} \to 0$ (Another notation: $f \ll g$)
(5) $f = \omega(g)$ if $\frac{g(n)}{f(n)} \to 0$ (Another notation: $f \gg g$)

Definition 3.5

Let A be an algorithm and T be its running time. We say that the running time of A is $O(f(n))$ if $T(n) = O(f(n))$.

If $\exists K \in \mathbb{N} : T(n) = O(N^K)$, we say that A is a polynomial time algoritm.

When $K = 1$, then A is a linear-time algorithm.

Definition 3.6 **Decision problem**

Decision problem: Problems that have a *YES* or *NO* answer.
Formally, this is a boolean function $f : X \to \{0,1\}$

Definition 3.7 **P-class**

A decision problem $f : X \to \{0,1\}$ is in the class P if $\exists$ algorithm that $\forall x \in X$ outputs $f(x)$ in time polynomial in $|X|$.

Example

$f : \mathbb{N} \to \{0,1\}$ given by $f(x) = \begin{cases} 1 & \text{, if x is prime} \\ 0 & \text{, otherwise} \end{cases}$

Definition 3.8 **NP-class**

A decision problem $f : X \to \{0,1\}$ is in NP if $\exists$ a verifier $V \in P$ and a polynomial p so that $\forall x \in X$:

$f(x) = 1 \iff \exists$ witness w with $|w| \leq p(|x|)$ and $V(x,w) = 1$

Lecture 05 (2025/10/27)

Example **Hamiltonicity**

$f : \{\text{All graphs}\} \to \{0,1\}; f(G) = \begin{cases} 1 & \text{if } G \text{ is Hamiltonian} \\ 0 & \text{otherwise} \end{cases}$

$f \in \text{NP}$

Definition 3.9 **co-NP-class**

A decision problem $f : X \to \{0,1\}$ is in co-NP if $-f + 1 \in \text{NP}$

Observation: $\text{P} \subseteq \text{NP} \cap \text{co-NP}$

Definition 3.10 **NP-hard**

A decision problem $f : X \to \{0,1\}$ is NP-hard if $\forall g \in \text{NP} : \exists$ polynomial time mapping $r \in \{\text{dom}(g) \to X\} : \forall x \in \text{dom}(g) : g(x) = 1 \iff f(r(x)) = 1$

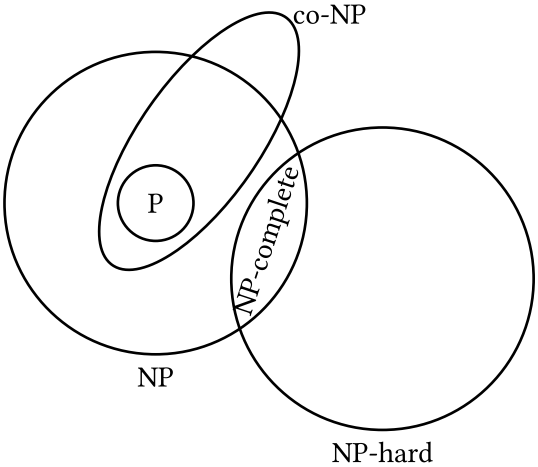
Definition 3.11 **NP-complete**

$\text{NP-complete} = \text{NP-hard} \cap \text{NP}$

Example **Clique problem is NP-complete**

Let G be a graph on $n \in \mathbb{N}$ vertices. The decision problem $f : \{\text{All graphs}\} \to \{0,1\}$ given by $f(G) = \begin{cases} 1 & \text{if } G \text{ has a clique of size } > 2\log n \\ 0 & \text{otherwise} \end{cases}$ is NP-complete.

Not known: Is P=NP and is NP=co-NP?



Algorithm 3.12 **GRAPH SCANNING ALGORITHM (GSA)**

Input: A graph G and $s \in V(G)$

Set $i = 0$, $R_0 := \{s\}$, $Q_0 := \{s\}$ and $T_0 := \emptyset$
while $Q_0 \neq \emptyset$, do:

Choose $v \in Q_i$:
If $R_i^c \cap N(V) \neq \emptyset$, then:

choose $w \in R_i^c \cap N(v)$
set $R_{i+1} := R_i \cup \{w\}$, $Q_{i+1} = Q_i \cup \{w\}$ and $T_{i+1} := T_i \cup \{vw\}$

else: set $R_{i+1} := R_i$, $Q_{i+1} := Q_i \setminus \{v\}$ and $T_{i+1} := T_i$
$i \leftarrow i + 1$

Output: (R_i, T_i)

Theorem 3.13

The GSA outputs a spanning tree of the connected component of G containing s.

Proof:
Claim 1: $Q_i \subseteq R_i \subseteq R_{i+1} \subseteq V(G)$ and $T_i \subseteq T_{i+1} \subseteq E(G) \forall i$

Induction on i and the two lines with "set R_{i+1}" gives that Claim 1 holds.

Claim 2: The algoritm works correctly.

As $2|R_i| - |Q_i| => 0$ for all i with $Q_i \neq \emptyset$, the algorithm must terminate.

Claim 3: (R_i, T_i) is a tree for all i.

Done by induction on i.

It is left to proof that for $H := (R, T) = \text{output of GSA(G,s)}, C_s := \text{connected component of s in G}, H = C_s$.

$R \subseteq C_s$ is clear by construction.

Suppose for contradiction that $\exists w \in C_s \setminus R$

Let $P = s\text{-}w$ path in G.

- $\exists i : R_{i+1} = R_i \cup \{x\}$
  $\Rightarrow Q_{i+1} = Q_i \cup \{x\}$

As GSA stops $\Rightarrow \exists T : Q_T = \emptyset$
$\Rightarrow \exists j : Q_{j+1} = Q_j \setminus \{x\}$
$\Rightarrow N(x) \cap R_j^c = \emptyset \Rightarrow N(x) \subseteq R_j$
$\Rightarrow y \in R_j \subseteq R$ which is a contradiction. □

Lecture 06 (2025/10/28)

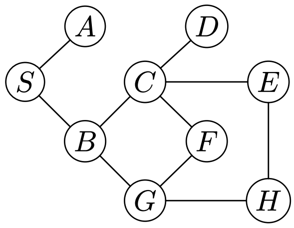
Observation: GSA runs in time $O(nm)$ where $n = v(G)$ and $m = e(G)$.

> **Algorithm 3.14** — **Breadth-First Search (BFS)**
>
> The GSA where the data structure of Q is a queue (-> First in - first out principle).

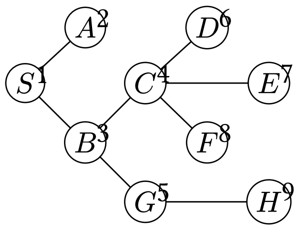
> **Algorithm 3.15** — **Depth-First Search (DFS)**
>
> The GSA where the data structure of Q is a stack (-> Last in - first out principle).

In GSA, Q_i is a vector and $Q_{i+1} = (Q_i, w)$
In BFS, in the first line: $v = Q_i[i]$
In DFS: in the first line: $v = Q_i[l]$ with $l = |Q_i|$

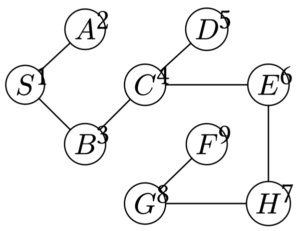
Example



BFS visits in the following order and edges:



DFS visits in the following order and edges:



Lemma 3.16

Let G be a connected graph and $s \in V(G)$.

If T is the tree output by BFS(G, s), then $d_T(s, u) = d_G(s, u) \forall u \in V(G)$

Let's add a new function in the BFS algorithm:

Algorithm **BFS with Distance**

Input: A graph G and $s \in V(G)$

Set $i = 0$, $R_0 := \{s\}$, $Q_0 := \{s\}$ and $T_0 := \emptyset$, $l(s) = 0$
while $Q_0 \neq \emptyset$, do:

Choose $v = Q_i[1]$
If $R_i^c \cap N(V) \neq \emptyset$, then:

choose $w \in R_i^c \cap N(v)$
set $R_{i+1} := R_i \cup \{w\}$, $Q_{i+1} = Q_i \cup \{w\}$ and $T_{i+1} := T_i \cup \{vw\}$ and $l(w) = l(v) + 1$

else: set $R_{i+1} := R_i$, $Q_{i+1} := Q_i \setminus \{v\}$ and $T_{i+1} := T_i$
$i \leftarrow i + 1$

Output: (R_i, T_i)

Proposition 3.17

The function l in the BFS with Distance algorithm satisfies is well-defined.

Proof of Proposition 3.17: $\forall w \in V(G), \exists i : R_{i+1} = R_i \cup \{w\}$
$\Rightarrow w \in R_j \forall j \geq i+1 \Rightarrow w \notin R_j^c \quad \forall j \geq i+1$ $\square$

Proposition 3.18

$l(v) = d_{R_i,T_i}(s,v)\ \forall v \in R_i \forall i$

Proof of Proposition 3.18: Induction on i.

Base case: $i = 0$. Then, $R_0 = \{s\}$ and $l(s) = 0 = d_{(R_0,T_0)}(s,s)$.

Induction hypothesis: Let $k \in \mathbb{N}$ and assume that $l(v) = d_{(R_k,T_k)}(s,v)\ \forall v \in R_k$.

Induction step: Let $v \in R_{k+1}$.

- If $v \in R_k$, then done by induction hypothesis.
- If $v \in R_{k+1} \setminus R_k$, then $\exists u \in R_k$ with $uv \in T_{k+1}$ and therefore

$d_{(R_{k+1},T_{k+1})}(s,v) = d_{(R_k,T_k)}(s,u) + 1 = l(u) + 1 = l(v)$ $\square$

Proposition 3.19

$l(w) \leq l(v_i) + 1 \forall w \in R_i$

Proof of Proposition 3.19 : Note that $l(v_j) \leq l(v_{j+1}) \forall j$

$w \in R_i \Rightarrow \exists j \leq i : l(w) = l(v_j) + 1 \leq l(v_i) + 1$ $\square$

Proof of Lemma 3.16: $d_T(s,u) \geq d_G(s,u)$ follows from $T \subseteq G$.

Suppose for a contradiction that $B := \{x : d_G(s,x) < d_T(s,x)\} \neq \emptyset$

Take $w \in B$ so that $D_G(s,w) = \min_{x \in B} d_G(s,x) =: d$

Let P be the shortest s-w path in G. Let v be the vertex before w on P.

Then, $d_G(v,s) = d - 1$
$\Rightarrow v \notin B$ (because of the minimality of d)
$\Rightarrow d_T(s,v) = d_G(s,v)$

$\Rightarrow s \notin T$ (otherwise $d_G(s,w) = d_T(s,w)$)

$l(w) = d_T(s,w) > d_G(s,w) = D_G(v,s) + 1 = d_T(v,s) + 1 = l(v) + 1$

$\Rightarrow l(w) > l(v) + 1$

$l(w) \leq l(v_i) + 1$ with $v_i = v$.

Contradiction. $\square$

Lemma 3.20

Let G be a connected graph and $s \in V(G)$.

If T is the tree output by DFS(G,s), then $\forall uv \in E(G) \exists$ a s-to-leaf path $P \subseteq T : u, v \in V(P)$

Claim: Q_i is a path $\forall i$
Suppose $i = \min\{t : v \in Q_t\} < \min\{t : u \in Q_t\}$
Let $j = \max\{t : v \in Q_t\}$

$\Rightarrow v_j = v$ and $Q_{j+1} = Q_j \setminus \{v\}$
$\Rightarrow N(v) \cap R_j^c = \emptyset \Rightarrow N(V) \subseteq R_j$
$$U \in R_j$$

$\Rightarrow \exists K \in \{i, ..., j\} : u \in Q_k$
$\Rightarrow u, v \in Q_k$
But Q_k is a path!

Lecture 07 (2025/11/03)

3.1 Minimum spanning trees

Definition 3.21 — **Minimum spanning tree**

A **minimum spanning tree** of a graph G with edge costs $c : E(G) \to \mathbb{R}$ is a spanning tree $T \subseteq G$ such that the total edge cost $c(T) := \sum_{e \in E(T)} c(e)$ is minimized.

Q/ G graph and $c : E(G) \to \mathbb{R}$. Is there a polynomial time algorithm to find the minimum spanning tree $T \subseteq G$?

Algorithm 3.22 — **Kruskal's algorithm**

Input: A graph G, $c : E(G) : \mathbb{R}$.

Sort the edges so that $c(e_1) \leq ... \leq c(e_m)$ (with $m := e(G)$).
Set $i = 0$ and $T_i = (V(G), \emptyset)$.
while $i < m$, do:

If $T_i + e_{i+1} \supseteq$ cycle, then

Set $T_{i+1} = T_i + e_i$

Else set $T_{i+1} = T_i$
$i \leftarrow i + 1$

Output: T_m

Theorem 3.23

Kruskal's algorithm outputs a spanning tree of minimum weight and its running time is $O(m^2)$

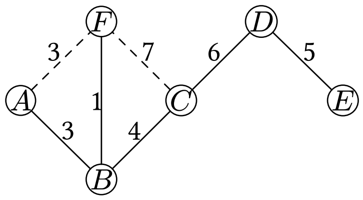
Definition 3.24

We'll write T_{xy} for the unique x-y path in a tree T

Lemma 3.25

T is a minimum spanning tree in $G \iff \forall xy \in E(G) \setminus E(T)\ \forall u, v \in T_{xy} : c(uv) \leq c(xy)$

Example



Proof of Theorem 3.23 assuming Lemma 3.25:
Let T be the tree output by Kruskal's Algorithm.
Let T^* be a minimum spanning tree (MST).
We assume $|E(T) \cap E(T^*)|$ is maximized among all MSTs.

Suppose for contradiction that T is not an MST, i.e., $c(T) > c(T^*)$.
$\Rightarrow T \neq T^* \Rightarrow \exists e \in E(T) \setminus E(T^*)$.

Among such edges, let e be the first one added by Kruskal's algorithm.
Consider $T^* + e$. This graph contains a unique cycle C.

Since T contains no cycles (it is a tree), there must be an edge $f \in C$ such that $f \notin E(T)$.

Claim: $c(e) \leq c(f)$.

Kruskal's algorithm considers edges in non-decreasing order of weight.
Since $e \in E(T)$ and $f \notin E(T)$, Kruskal considered e before f (or at the same time).
If $c(f) < c(e)$, Kruskal would have considered f first. Since f is part of $C \subseteq T^* + e$, adding f earlier would not have created a cycle in the subgraph of T constructed so far (as e wasn't added yet). Thus f would have been added to T. Contradiction.

Now consider $T^{**} = T^* + e - f$.
This is a spanning tree because we broke the cycle C but kept connectivity.

$c(T^{**}) = c(T^*) + c(e) - c(f) \leq c(T^*)$.

Since T^* is an MST, $c(T^{**})$ cannot be strictly smaller, so $c(T^{**}) = c(T^*)$. Thus, T^{**} is also an MST.

However, $E(T^{**})$ contains e (which is in T) and all edges of $T^* \cap T$. $\Rightarrow |E(T^{**}) \cap E(T)| = |E(T^*) \cap E(T)| + 1$.

This contradicts the assumption that T^* maximized the intersection with T. Therefore, T must be a minimum spanning tree. $\square$

Proof of Lemma 3.25:
$\Longrightarrow$:
Suppose for contradiction that $\exists xy \in E(G) \setminus E(T)$ and $e \in T_{xy} : c(e) > c(xy)$

Define $T' = T - e + xy$. $\Rightarrow c(T') < c(T)$

Goal: T' is a spanning tree.

$T - e$ has two connected components C_x and C_y
$\Rightarrow T'$ is connected and $e(T') = e(T) = v(T) - 1$
$\Rightarrow T'$ is a tree

$\Longleftarrow$:
Let $T \subseteq G$ be a tree such that $\forall xy \in E(G) \setminus E(T)\ \forall u, v \in T_{xy} : c(uv) \leq c(xy)$.

Let T' be a minimum spanning tree of G with $|E(T') \cap E(T)| = \max\{|E(T'')| \cap |E(T)| :$
T'' is a minimum spanning tree of $G\}$

Suppose for contradiction that $T \neq T'$.
$\Rightarrow \exists xy \in E(T) \setminus E(T')$.

$T' - xy$ has two components C_x and C_y

$\exists e \in T_{xy} : |e \cap C_x| = |e \cap C_y| = 1$

$\Rightarrow c(e) \leq c(xy)$

Let $T'' = T' - xy + e$
T'' is a spanning tree as T' is a spanning tree ($\Rightarrow e(T'') = e(T') = v(T'') - 1$ and T'' is connected)

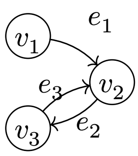
$|E(T'') \cap E(T)| > |E(T') \cap E(T)|$ which is a contradiction. $\square$

> **Note** **Runtime of Kruskal's algorithm**
>
> To check that cycle $\nsubseteq T_i + e_{i+1}$ requires $O(m)$ (Run DFS and BFS for example).
>
> This is done for every $i \in \{1, ..., m\}$.
>
> The total running time is therefore $O(m^2)$

4 Euler tours and Hamiltonian cycles

Recall 4.1

A **walk** is an alternating sequence of vertices and edges $v_1 e_{21} v_2 e_2 ... v_k e_k v_{k+1}$ so that $e_i = v_i v_{i+1} \forall i \in \{1, ..., k\}, k \in \mathbb{N}$.



$= v_1 e_1 v_2 e_2 v_3 e_3 v_2$

k is called the **length** of the walk.

Definition 4.2

A walk is called **closed** if $v_1 = v_k + 1$

A walk is **simple** if $e_i \neq e_j \forall i \neq j$

Definition 4.3

Let $W = v_1 e_1 v_2 e_2 ... v_k e_k v_{k+1}$ be a walk.

$V(W) := \{v_1, v_2, ..., v_{k+1}\}$,
$E(W) := \{e_1, e_2, ..., e_k\}$
$G_W = (V(W), E(W))$

Definition 4.4 **Euler Tour**

A walk W in a graph G ($E(W) \subseteq E(G)$) is an **Euler tour** if W is closed, simple and $G_W = G$.

Definition 4.5 **Eulerian graph**

An **Eulerian** (graph) is a graph G in which there exists an Euler tour in it.

Theorem 4.6

A connected graph G is Eulerian $\iff d_G(v) \equiv 0 \pmod 2 \ \forall v \in V(G)$

Proof:
Claim 1:
Let W be a simple and closed walk in G. Then $d_{G_w} \equiv 0 \pmod 2 \ \forall v \in V(G)$.

$d_{G_w} = 2 \cdot \#\{i : v_i = v\}$

#

Claim 2:
Let G be a graph with $d_G(v) \equiv 0 \pmod 2 \ \forall v \in V(G)$.
Then every maximal simple walk in G is closed.

Let $W = v_1 e_1 ... v_k e_k v_{k+1}$ be a maximal simple walk in G. Suppose for contradiction that $v_1 \neq v_{k+1}$
$\Rightarrow d_{G_w}(v_1) \equiv 1 \pmod 2$.

Since $d_G(v_1) \equiv 0 \pmod 2$, we could have extended this walk by another edge, which is a contradiction.

#

$\Longrightarrow$: follows directly from Claim 1.

$\Longleftarrow$: Let W be a maximum simple walk in G. By Claim 2, W is closed.

Suppose for contradiction that $G_W \neq G$.

As G is connected and $G_W \neq G$, $\exists uv \in E(G) \setminus E(G_W) : u \in V(G_W)$

$G - G_W$ is a graph with $d_{G-G_W}(v) \equiv 0 \pmod 2$ (as G has even degree for every vertex and by Claim 1 G_W has even degree for every vertex)

By Claim 2, there is a maximal simple and closed walk W' in $G - G_W$ starting at u.

$\Rightarrow E(G_{W'}) \neq \emptyset$

Take a walk W'' in G_W starting at u such that $G_{W''} = G_W$ (by doing a circular shift of W).

Then $W'W''$ is a longer walk, wich is a contradiction □

Lecture 08 (2025/11/04)

Definition 4.7 **Hamiltonian cycles**

A cycle $C \subseteq G$ is **Hamiltonian** if $V(C) = V(G)$.

We say that G is Hamiltonian if it contains a Hamiltonian cycle.

Theorem 4.8 **Dirac '52**

$\delta(G) \geq \frac{v(G)}{2} > 1 \Rightarrow G$ is Hamiltonian

Proof:
Claim: G is connected

Suppose not, then $\exists$ connected component $C \subseteq V(G)$ with $v(C) \leq \frac{v(G)}{2}$
$\Rightarrow \exists$ vertex v in $C : d_G(v) \leq v(C) - 1 < \frac{v(G)}{2}$, which is a contradiction

"Claim"

Let $P = v_0 v_1 ... v_k$ be a maximal path in G
$\Rightarrow N_G(v_0) \subseteq V(P); N_G(v_k) \subseteq V(P)$ (otherwise it could be extended)

$N_0 := \{i \in \{1, ..., k\} : v_0 \sim v_i\}$
$N_k := \{i \in \{0, ..., k-1\} : v_k \sim v_i\}$

We define a notation: $A - x := \{a - x \mid a \in A\}$ for sets A.

Observation: $N_k \cap (N_0 - 1) \neq \emptyset \Rightarrow G[V(P)]$ is Hamiltonian (because if $i \in N_k \cap (N_0 - 1)$, then the path $v_0 \to v_{i+1} \to v_{i+2} \to ... \to v_k \to v_i \to v_{i-1} \to ... \to v_0$ is a Hamiltonian cycle)

We know: $N_k \subseteq \{0, ..., k-1\}, N_0 \subseteq \{0, ..., k-1\}$

$$|N_k \cap (N_0 - 1)| = |N_k| + |N_0 - 1| - |N_k \cup (N_0 - 1)|$$
$$\geq \frac{v(G)}{2} + \frac{v(G)}{2} - k \geq v(G) - k \geq 1 \quad \text{(as Path } P \text{ already has } k+1 \text{ vertices)}$$

Therefore $G[V(P)]$ is Hamiltonian.

If $V(P) = V(G)$, then we are done.

Suppose for contradiction that $V(P) \neq V(G)$.
As the graph is connected, $\exists u \in V(P), w \in V(G) \setminus V(P) : u \sim w$. We showed that P is a cycle, we can therefore break the cycle at u and extend it to w, which is a contradiction to P being maximal. □

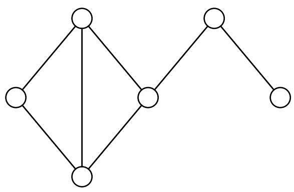
Definition 4.9 **degree sequence**

For a graph on n vertices with degrees $d_1 \leq \ldots \leq d_n$, we call $s_G = (d_1, \ldots, d_n)$ the **degree sequence** of G.

Note

For a given graph the degree sequence is unique. For a given degree sequence the graph is generally not unique.

Example



has degree sequence $(1, 2, 2, 3, 3, 3)$

C_6 and $C_3 \cup C_3$ both have degree sequence $(2, 2, 2, 2, 2, 2)$

Definition 4.10

Notation: $(x_1, \ldots, x_n) \leq (y_1, \ldots, y_n) :\Longleftrightarrow \forall i \in \{1, \ldots, n\} : x_i \leq y_i$

Definition 4.11

$(a_1, \ldots, a_n)$ is Hamiltonian if $\forall$ graph $G : s_G \geq (a_1, \ldots, a_n) \Rightarrow G$ is Hamiltonian

Note

Using Dirac's theorem it follows that $\underbrace{\left(\frac{n}{2}, \ldots, \frac{n}{2}\right)}_{n \text{ times}}$ is Hamiltonian

Theorem 4.12 **Chvátal '72**

Let $n \geq 3$ and $0 \leq a_1 \leq \ldots \leq a_n \leq n$ (with $\forall i : a_i \in \mathbb{Z}$)

$(a_1, \ldots, a_n)$ is Hamiltonian $\Longleftrightarrow \forall i < \frac{n}{2} : a_i \leq i \Rightarrow a_{n-i} \geq n - i$

Proof:

$\Longrightarrow$:

Suppose for contradiction that $\exists i < \frac{n}{2} : a_i \leq i \wedge a_{n-i} \leq n-i-1$ (with $(a_1, ..., a_n)$ Hamiltonian)

Goal: to find a non-Hamiltonian G with $s_G \geq r \geq (a_1, ..., a_n)$ (with r being a degree sequence)

Observation: $r = \left(\underbrace{i, ..., i}_{i \text{ times}}, n-i-1, ..., \underbrace{n-i-1}_{\text{entry } n-1}, n-1, ..., n-1\right)$

We can construct such a graph G by talking an independent set of i vertices ($=: B$). Then, we take a complete graph on $n-i$ vertices ($=: C$). Finally, we take i vertices from the complete graph C ($=: A$) and connect all of them with all of the i vertices in the independent set B.

Claim 1: $G - A$ has $i+1$ connected components. (the i vertices from B and $C - A$)

Claim 2: If $I \subseteq V(H)$ with H a Hamiltonian Cycle, $|I| = i$, then $\#$ connected components in $H - I \leq i$ (the proof is done via induction)

Claim 1+2 gives us a contradiction because if G was Hamiltonian and we remove A from it, we would have at most i connected components, but Claim 1 states that we have $i+1$ connected components.

"⇒"

$\Longleftarrow$:

Let $(a_1, ..., a_n)$ be a degree sequence satisfying $\forall i < \frac{n}{2} : a_i \leq i \Rightarrow a_{n-i} \geq n-i$.
Suppose for a contradiction that $(a_1, ..., a_n)$ is not Hamiltonian.

Let $B := \{G \text{ non-Hamiltonian} \mid s_G \geq (a_1, ..., a_n)\} \neq \emptyset$

Take $G \in B$ with $e(G) = \max\{e(H) \mid H \in B\}$.

Claim: for a graph H with $s_H = (d_1, ..., d_n) \geq (a_1, ..., a_n) \Rightarrow s_H$ satisfies $\forall i < \frac{n}{2} : d_i \leq i \Rightarrow d_{n-i} \geq n-i$

Let $i < \frac{n}{2}$ with $d_i \leq i$. Then, $a_i \leq d_i \leq i \Rightarrow d_{n-1} \geq a_{n-1} \geq n-1$

"Claim"

$\mathcal{P} := \{(u, v) \mid u \neq v, uv \notin E(G), d_G(u) \leq d_G(v)\}$.

Take $(x, y) \in \mathcal{P}$ with $d_G(x) + d_G(y) = \max\{d_G(u) + d_G(v) \mid (u, v) \in \mathcal{P}\}$.

For simplicity, $r := d_G(x)$

Lemma **Lemma 1**

$d(x) + d(y) \leq n-1$

Lemma **Lemma 2**

$\#\{v : d_G(v) \leq r\} \geq r$

Let $s_G =: (d_1, ..., d_n)$.

We know $r < \frac{n}{2}$ by assumtion and by Lemma 2: $d_r \leq r$
$\Rightarrow d_{n-r} \geq n - r$
$\Rightarrow n - r \leq \underbrace{d_{n-r} \leq d_{n-r+1} \leq \ldots \leq d_n}_{r+1 \text{ times}}$

$d_G(x) = r \Rightarrow \exists z \in V(G) : d_G(z) \geq n - r \wedge z \nsim x$
$\Longrightarrow n - 1 \overset{\text{Lm } 1}{\geq} d_G(x) + d_{G(y)} \geq d_G(x) + d_G(z) \geq n - r + r = n$ □

Lecture 09 (2025/11/10)

Proof of Lemma 1:
$s_{G+xy} \geq s_G \geq \vec{a}$

$\Rightarrow G + xy$ is Hamiltonian

$\Rightarrow \exists$ path $P = v_0 v_1 \ldots v_{n-1} \subseteq G$ with $v_1 = x, v_{n-1} = y$

Let $N_0 := \{i \mid v_0 \sim v_i\}$ and $N_{n-1} = \{i \mid v_{n-1} \sim v_i\}$

Observation 1: $(N_0 - 1) \cap N_{n-1} = \emptyset$

Observation 2: $N_0 - 1, N_{n-1} \subseteq \{0, \ldots, n-2\}$

$d_G(x) + d_G(y) = |N_0 - 1| + |N_{n-1}| \overset{\text{Obs1}}{=} |(N_0 - 1) \cup N_{n-1}| \overset{\text{Obs2}}{\leq} n - 1$ □

Proof of Lemma 2:
Let $i \in N_0 - 1$.
If $v_i \sim y$, then (as seen in Lemma 1) we find a Hamiltonian cycle, which is a contradiction.
$\Rightarrow v_i \nsim y$.

By the maximality of $d_G(x) + d_G(y)$ among non-adjacent pairs, we must have:
$d_G(v_i) + d_G(y) \leq d_G(x) + d_G(y)$
$\Rightarrow d_G(v_i) \leq d_G(x) = r$.

The set of vertices $\{v_i : i \in N_0 - 1\}$ has size $|N_0 - 1| = |N_0| = d_G(x) = r$.
Thus, we have found at least r vertices with degree $\leq r$. □

5 Travelling salesperson problem

Definition 5.1 **Travelling salesperson problem (TSP)**

A **Travelling salesperson problem (TSP)** is: Given a complete graph K_n and $c : E(K_n) \to \mathbb{R}_{>0}$, find a Hamiltonian cycle $C \subseteq K_n$ whose weight $c(C) := \sum_{e \in E(C)} c(e)$ is minimized.

A **Metric TSP** is a c such that $c(xy) \leq c(xz) + c(zy)$

$\mathrm{OPT}(K_n, c) := \operatorname{argmin}\{c(C) \mid C \subseteq K_n, C \text{ is Hamiltonian}\}$

Definition 5.2

$\boldsymbol{A}$ is a **$\boldsymbol{\rho}$-approximation algorithm** for $f : I \to \mathbb{R}_{\geq 0}$ (with $\rho \in \mathbb{R}$) if $\boldsymbol{A}(x) \leq p \cdot f(x) \forall x$ and $\boldsymbol{A}$ is a polynomial time algorithm.

Definition 5.3

$\mathrm{MST}(K_n, c) := \operatorname{argmin}\{c(T) \mid \text{spanning tree } T \subseteq K_n\}$

Lemma 5.4

Given a TSP instance (K_n, c), we have $c(\mathrm{MST}(K_n, c)) \leq c(\mathrm{OPT}(K_n, c))$

Proof: Let $C = \mathrm{OPT}(K_n, c)$

Take $e \in E(C)$. $C - e$ is a tree.

Therefore, $c(\mathrm{MST}(K_n, c)) \leq c(C - e) \leq c(C)$ □

Definition 5.5 **Multigraph**

A multigraph (V, E) is a pair where E is a multiset sucht that $\forall e \in E : e \subseteq V \land |e| = 2$.

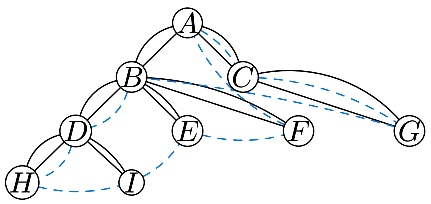
Algorithm 5.6 **double tree algorithm (DTA)**

Input: K_n and $c : K_n \to \mathbb{R}_{>0}$

1 Find a MST T
2 Duplicate T's edges into a multigraph
3 Take an Eulerian tour W in T
4 Shortcut (by skipping all doubled vertices) W to obtain a Hamiltonian cycle C

Output: C

Example



$W = ACGCABDHDIDBEBFBA$

$C = ACGBDHIEFA$ (blue)

Lemma 5.7

In metric TSP, the double tree algorithm satisfies
$C = \mathrm{DTA}(K_n, c) \Rightarrow c(C) \leq 2c(\mathrm{MST}(K_n, c))$

Proof: The proof follows from the triangle inequality and the fact that every edge in the MST is traversed exactly twice in the Eulerian tour. □

Corollary 5.8

DTA is a 2-approximation algorithm for Metric TSP.

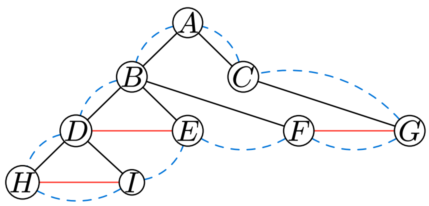
Algorithm 5.9 **Christofides' algorithm**

Input: K_n and $c : K_n \to \mathbb{R}_{>0}$ metric

1 Find a MST T
2 $\sigma := \{v \mid d_T(v) \equiv 1 (\operatorname{mod} 2)\}$
3 Find a minimum weight perfect matching (a 1-regular graph; exists as $|\sigma|$ is always odd) $M \subseteq K_n[\sigma]$ with respect to $cK_n \to \mathbb{R}_{>0}$ (running time $O(n^3)$ without proof)
4 $G := (V(K_n), E(T) \cup E(M))$ and take an Eulerian tour W in G
5 Shortcut (by skipping all doubled vertices) W to obtain a Hamiltonian cycle C

Output: C

Example



In red: the perfect matching M

$W = ABDHIDEBFGCA$

$C = ABDHIEFGCA$ (blue)

Note

$c(C) \leq c(\text{MST}) + c(M)$

Lemma 5.10

For Metric TSP and the minimum weight perfect matching M on the odd degree vertices of a MST and the output C of Christofides' algorithm, we have
$c(M) \leq \frac{1}{2} c(\text{OPT}(K_n, c))$

$\Rightarrow c(C) \leq c(\text{MST}(K_n, c)) + c(M) \leq c(\text{OPT}(K_n, c)) + \frac{1}{2} c(\text{OPT}(K_n, c)) = \frac{3}{2} c(\text{OPT}(K_n, c))$

Proof: Let $D = \text{OPT}(K_n, c))$ Hamiltonian cycle of min. weight

Let C be a Hamiltonian cycle in $K_{n[\sigma]}$

Let M be a minimum weight perfect matching in $K_{n[\sigma]}$

We can split C into two matchings M_1 and M_2 (by taking every second edge)

Then: $c(M) \leq \frac{c(M_1)+c(M_2)}{2} \leq \frac{c(C)}{2} \leq \frac{c(D)}{2}$ □

Lecture 10 (2025/11/11)

6 Bipartite Graphs and matchings

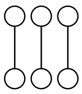
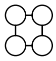
Definition 6.1

Let G be a graph. G is **bipartite** if $\exists A, B \subseteq V(G) : V(G) = A \cup B$ and $e(G[A]) = e(G[B]) = 0$.

We write $G[A, B]$ to denote that G is a bipartite graph with bipartition (A, B).

Example

Bipartite graphs:

$\emptyset$, , , even cycles, $K_{n,m} := (A \cup B, \{ab \mid a \in A, b \in B\})$ (with $|A| = n, |B| = m$)

Non-bipartite graphs: complete graph on ≥ 3 vertices, odd cycles

Definition 6.2

For two vertex sets A, B we define $K[A, B]$ as the complete bipartite graph with bipartition (A, B): $K[A, B] := (A \cup B, \{ab \mid a \in A, b \in B\})$

Lemma 6.3

Let G be a graph. Then,

$$G \text{ bipartite} \iff \text{odd cycle} \not\subseteq G$$

Proof:

$\Longleftarrow$:

W.L.O.G. assume G is connected. (otherwise apply the argument to each connected component)
Let $T \subseteq G$ be a spanning tree of G.

Claim: T is bipartite

Let $r \in V(T)$.
Let $c : V(T) \to \{0, 1\}$ with $c(v) := d_T(r, v) \bmod 2$.

$$\forall uv \in E(T) : |d_T(u, r) - d_T(v, r)| = 1$$
$$\Rightarrow c(u) \neq c(v)$$

Therefore, T is bipartite with $(A, B) := (\{v \in V(T) \mid c(v) = 0\}, \{v \in V(T) \mid c(v) = 1\})$

#

Let (A, B) be the bipartition of T.

Claim: G is bipartite with bipartition (A, B)

Suppose for contradiction that $\exists uv \in E(G[A])$

Let P be the unique path in T between u and v.
Then, $P + uv$ is an odd cycle in G (as $uv \notin E(T)$ as it is bipartite with (A, B)), which is a contradiction.

$\Longrightarrow$:
Let G be a bipartite graph with bipartition (A, B).
Suppose for contradiction that $\exists$ odd cycle $C \subseteq G$.
Let $v_0 v_1 ... v_k v_0$ be the odd cycle (with k odd). W.L.O.G. assume $v_0 \in A$
$\Rightarrow v_1 \in B, v_2 \in A, ..., v_k \in B, v_0 \in A$
This is a contradiction. $\square$

Definition 6.4 **matching**

M is a **matching** if $d_M(v) = 1 \quad \forall\ v \in V(M)$.

We call $e(M)$ the **size** of the matching.

The size of a maximum matching is called the **matching number** and is denoted by $\nu(G)$.

Q/ What is the size of the largest matching we can find in a bipartite graph?

Definition 6.5 **vertex cover**

Let G be a graph. $X \subseteq V(G)$ is a **vertex cover** if $\forall e \in E(G) : e \cap X \neq \emptyset$

Note

For a matching M in G and $X \subseteq V(G)$ a vertex cover,

$$|M| \leq |X|$$

Theorem 6.6 **König**

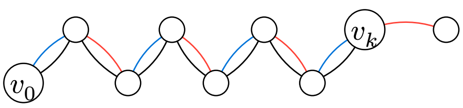
Let G be a bipartite graph, then

$$\max\{e(M) \mid M \subseteq G \text{ matching}\} = \min\{|X| \mid X \subseteq V(G) \text{ vertex cover}\}$$

Definition 6.7 **M-alternating path**

Let M be a matching. A path $P = v_0 v_1 ... v_k$ is an ***M*-alternating path** if $v_0 \notin V(M)$ and $P - E(M)$ is a matching.

Example



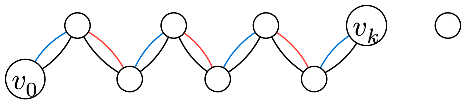
Matching M is marked red. The path P (marked black) is an M-alternating path because $P - E(M)$ (marked blue) is a matching and $v_0 \notin V(M)$.

Definition 6.8 **M-augmenting path**

Let G be a graph and $M \subseteq G$ be a matching.

A path $P = v_0 v_1 ... v_k$ is ***M*-augmenting** if P is M-alternating and $v_k \notin V(M)$

Example



Matching M is marked red. The path P (marked black) is an M-alternating path because $P - E(M)$ (marked blue) is a matching and $v_0 \notin V(M)$ and $v_k \notin V(M)$

Note

As the name suggests, an M-augmenting path $P = v_0 v_1 ... v_k$ can be used to augment a matching M:

$$M' := (V(M), E(M) \Delta E(P))$$

(toggle all edges along the augmenting path)

Then M' is a matching and its size increases by one:

$$e(M') = e(M) - |E(P) \cap M| + |E(P) \setminus M| = e(M) - t + (t + 1) = e(M) + 1$$

since along an M-augmenting path edges alternate and there is exactly one more non-M edge than M-edge.

Proof of Theorem 6.6:
Let $\nu(G) := \max\{e(M) \mid M \subseteq G \text{ matching}\}$ and $\tau(G) := \min\{|X| \mid X \subseteq V(G) \text{ vertex cover}\}$.
We need to show that $\nu(G) = \tau(G)$.

We will show that there is a vertex cover of size $\nu(G)$.

Let (A, B) be the bipartition of G and M a maximal matching.

Let $f : E(M) \to V(M); uv \mapsto \begin{cases} v \text{ if } \exists x {\in} A \exists x {\to} v \ M\text{-alternating path} \\ u \text{ otherwise} \end{cases}$

Let $U = f(E(M))$. U has size $|U| = |E(M)| = \nu(G)$. We will now show that U is a vertex cover.

Let $ab \in E(G) \setminus E(M)$ (where $a \in A$, $b \in B$).
If $a \in U$ then we're done.

Assume then that $a \notin U$ (we have to show $b \in U$)

If $a \notin V(M)$

, then ab is an M-alternating path (because $b \in V(M)$ or else ab would be an M-augmenting path $\Rightarrow M$ not maximal).
$\Rightarrow b \in U$.

If $a \in V(M)$

, then let c be the matched vertex to a in M.
By assumption, $a \notin U$.
$\Rightarrow f(ac) = c \Rightarrow \exists x \in A, x \to c$ M-alternating path P.

Extending this path to $x \xrightarrow{P} c \xrightarrow{ac} a \xrightarrow{ab} b$ gives an M-alternating path from $x \in A$ to b.

As above $b \in V(M)$ (or else $x \xrightarrow{P} c \xrightarrow{ac} a \xrightarrow{ab} b$ would be an M-augmenting path $\Rightarrow M$ not maximal). Therefore, there is an $c \in V(M)$ with $cb \in E(M)$.

$f(cb) = b$ (as there is the M-alternating path)

$\square$

Definition 6.9

Let G be a graph, $M \subseteq G$ a matching and $A \subseteq V(G)$.

$\boldsymbol{M}$ **covers** $\boldsymbol{A}$ if $A \subseteq V(M)$.

Definition 6.10 **Hall's condition**

For a bipartite graph $G[A, B]$. G satisfies **Hall's condition** (also called **arriage condition**) if $|N_G(S)| \geq |S|$ for all $S \subseteq A$.

Theorem 6.11 **Hall '35**

Let $G[A, B]$ be a bipartite graph. Then for $A \subseteq G$:

$$\exists \text{ a matching covering } A \text{ in } G \iff G \text{ satisfies Hall's condition} \iff |N_G(S)| \geq |S| \ \forall S \subseteq A$$

Lecture 11 (2025/11/17)

Proof:
$\Longrightarrow$: trivial

#

$\Longleftarrow$: through induction on $|A|$:
Base case: $|A| = 1$, then $|N_G(A)| \geq 1$ (because the matching has to include an edge to A)

Induction step:
Let $a \in \mathbb{N}$.
I.H.: $|X| \leq a - 1$ and $H[X, Y]$ satisfying Hall's condition $\Rightarrow H[X, Y] \supseteq$ matching covering X
Suppose I.H. holds.

Let $G[A, B]$ be any graph satisfying Hall's condition, $|A| = a$.

Claim: If $|N_G(A)| \geq |S| + 1 \quad \forall\ \emptyset \neq S \subsetneq A$, then the proof is done.

Let $ab \in E(G)$.
Then, $|N_H(S)| \geq |N_G(S)| - 1 \geq |S| \quad \forall\ \emptyset \neq S \subseteq A \setminus \{a\}$

$\Rightarrow H$ satisfies Hall's condition.
Using the I.H., the Claim is shown.

#

Assume (for contradiction) $\exists \emptyset \neq S \subsetneq A$ such that $|N_G(S)| = |S|$

Let $H_1[S, N_G(S)]$ and $H_2[A \setminus S, B \setminus N_G(S)]$.
$(\Rightarrow G = H_1 \cup H_2)$

Hall's condition is satisfied by H_1.
By I.H. (as $S \subsetneq A$), $\exists$ matching in H_1 covering S.

Claim: H_2 satisfies Hall's condition.

Suppose not.
$\exists T \subseteq A \setminus S$ such that $|N_{H_2}(T)| \leq |T| - 1$ (because

$\Rightarrow |N_G(T) \setminus N_G(S)| = |N_{H_2}(T)| \leq |T| - 1$

$\Rightarrow |N_G(S \cup T)| = |N_G(S)| + |N_G(T) \setminus N_G(S)| \leq |S| + |T| - 1 < |S \cup T|$

This contradicts the assumption that G satisfies Hall's condition (applied to $S \cup T \subseteq A$).

#

By I.H. (as $S \subsetneq A$), $\exists$ matching in H_2 covering $A \setminus S$.

The union of the two matchings covering S and $A \setminus S$ gives a matching covering A. □

6.1 Applications of Hall's theorem

Definition 6.12

$M \subseteq G$ is a **perfect matching in G** or a **1-factor** if M is a matching and $V(M) = V(G)$

Lemma 6.13

Any k-regular bipartite graph ($k \geq 1$) has a perfect matching

Proof:
Let $G[A, B]$ be a k-regular bipartite graph.

For $S \subseteq A$, we know: $k - |S| = e(G[S, N(S)]) \leq |N(S)| \cdot k$
$\Rightarrow |S| \leq |N(S)| \quad \forall\ S \subseteq A$
$\Rightarrow |A| \leq |B|$ (because we can take $S = A$)

By the same argument: $|S| \leq |N(S)| \quad \forall\ S \subseteq B \Rightarrow |B| \leq |A|$

$\Rightarrow |A| = |B|$

Therefore, G satisfies Hall's condition $\Rightarrow$ exists a matching covering A.

Since $|A| = |B|$, this matching is perfect. □

Definition 6.14

H is called a **k-factor** of G if H is k-regular and $V(H) = V(G)$

Lemma 6.15

Let G be a $2k$-regular graph (k >=1).
Then, G has a 2-factor.

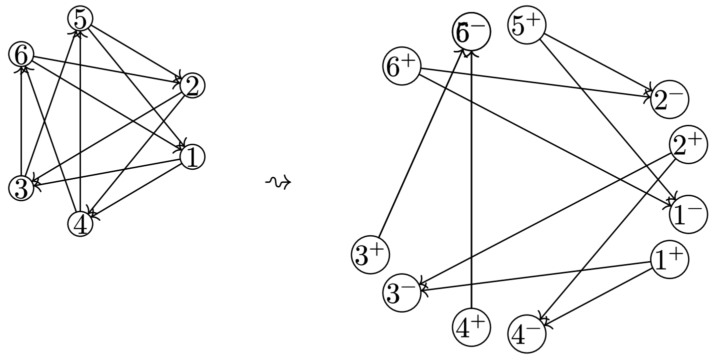
Proof:
G has an Eulerian tour. Let D be the associated digraph (graph with directed edges).

For each vertex $v \in V(D)$, split it into v^- and v^+ (just defining new vertices).

Define $H = (\{v^-, v^+ \mid v \in V(D)\}, \{u^+v^- \mid (u, v) \in E(D)\})$

Example

Let G be the following graph:



Then H is bipartite and k-regular.

By Lemma 6.13, H has a perfect matching.

This holds true for every G. H is bipartite and k-regular and therefore has a perfect matching. □

Q/ When does a graph have a perfect matching?

Definition 6.16

Notation:
For a graph H, let $q(H) := \#$ connected components in H with an odd number of vertices

Theorem 6.17 **Tutte**

Let G be a connected graph. Then,

$$G \text{ has a perfect matching} \iff \forall S \subseteq V(G), q(G-S) \leq |S|$$

Definition 6.18

A graph C is called **critical** if $C - v$ has a perfect matching $\forall v \in V(C)$.

Definition 6.19

Notation:
$\mathscr{C}_H := \{C \subseteq H \mid C \text{ is a connected component in } H\}$

Definition 6.20

$S \subseteq V(G)$ is **matchable** with $\mathscr{C}_{G-S}$ if the auxiliary graph H_S with $V(H_S) = S \cup \mathscr{C}_{G-S}$ and $E(H_S) = \{sC : s \in S, C \in \mathscr{C}_{G-S} \text{ and } N_G(s) \cap V(C) \neq \emptyset\}$ has a matching covering S.

Theorem 6.21 **Gallai-decomposition**

Let G be a connected graph. Then $\exists S^* \subseteq V(G)$ so that:
1. $\forall C \in \mathscr{C}_{G-S^*} : C$ is critical
2. S^* is matchable with $\mathscr{C}_{G-S^*}$

Moreover, G has a perfect matching $\iff |\mathscr{C}_{G-S^*}| = |S^*|$

Note

We can calculate the matching number $\nu(G)$ using the set S^*:

$$\nu(G) = \frac{1}{2}(v(G) - q(G - S^*) + |S^*|)$$

This follows from the fact that each odd component in $G - S^*$ (which are critical) leaves one vertex unmatched, and the vertices in S^* can be matched with $|S^*|$ of those unmatched vertices.

Lecture 12 (2025/11

Proof of Tutte's theorem assuming Gallai-decomposition theorem:

$\Longrightarrow$:
Let G have a perfect matching M. Let $S \subseteq V(G)$.

Let C be a component of $G - S$ with an odd number of vertices (C is an odd component).

Since $|V(C)|$ is odd and M is a perfect matching, the vertices in C cannot be fully matched among themselves (edges within C cover an even number of vertices). $\Rightarrow \exists v \in V(C) : v$ is matched to a vertex $u \in V(G) \setminus V(C)$.

Since C is a connected component in $G - S$, the only neighbors of C outside of C are in S. $\Rightarrow u \in S$.

Therefore, every odd component in $G - S$ requires at least one vertex in S to match with. Since M is a matching, these vertices in S must be distinct for each component.

$\Rightarrow |S| \geq q(G - S)$

$\Longleftarrow$:
Assume $q(G - S) \leq |S| \quad \forall\ S \subseteq V(G)$.

By the Gallai-decomposition theorem, $\exists S^* \subseteq V(G)$ such that:
1. All components in $\mathscr{C}_{G-S^*}$ are critical.
2. S^* is matchable with $\mathscr{C}_{G-S^*}$.

Step 1: Since every component $C \in \mathscr{C}_{G-S^*}$ is critical, $|V(C)|$ must be odd (as $C - v$ has a perfect matching, $v(C) - 1$ is even). $\Rightarrow |\mathscr{C}_{G-S^*}| = q(G - S^*)$

Step 2: Since S^* is matchable with $\mathscr{C}_{G-S^*}$, there exists a matching in the auxiliary graph covering S^*. $\Rightarrow |S^*| \leq |\mathscr{C}_{G-S^*}| = q(G - S^*)$

Step 3: By our initial assumption (Tutte's condition) applied to S^*: $q(G - S^*) \leq |S^*|$

Combining Steps 2 and 3, we get $|\mathscr{C}_{G-S^*}| = |S^*|$.

By the "Moreover" part of the Gallai-decomposition theorem: $|\mathscr{C}_{G-S^*}| = |S^*| \iff$ G has a perfect matching

Therefore, G has a perfect matching. $\square$

Proof of Gallai-decomposition theorem:
We proceed by induction on $v(G)$.

Base case: For $G = (\{a, b\}, \{ab\})$, $S = \emptyset$ works (check manually).

Induction hypothesis: Assume the theorem holds for any graph H with $v(H) < v(G)$. In particular, Tutte's theorem holds for any H with $v(H) < v(G)$.

Let $r(T) \coloneqq q(G - T) - |T|$ for any $T \subseteq V(G)$.

Let $\mathscr{R} \coloneqq \{T \subseteq V(G) \mid r(T) = \max_{T'} r(T')\}$.
Choose $S \in \mathscr{R}$ such that $|S|$ is maximized (i.e., $|S| = \max_{T \in \mathscr{R}} |T|$).

Claim 1: All connected components in $G - S$ are odd. (Implies $|\mathscr{C}_{G-S}| = q(G - S)$).

Suppose not. Let $C \in \mathscr{C}_{G-S}$ be a component of $G - S$ so that $|C|$ is even.
Let $v \in V(C)$.

Since $|V(C)|$ is even, $C - v$ has odd order, so $q(C - v) \geq 1$.

$$q(G - S - v) = q(G - S) - \underbrace{0}_{\text{C is even}} + q(C - v) \geq q(G - S) + 1$$

$\Rightarrow r(S \cup \{v\}) = q(G - S - v) - (|S| + 1) \geq q(G - S) + 1 - |S| - 1 = r(S)$.

This means $S \cup \{v\}$ also maximizes r, but $|S \cup \{v\}| > |S|$, contradicting the choice of S as the largest maximizer.

#

Proof of 1 (Criticality):

Suppose $\exists C \in \mathscr{C}_{G-S} : C$ not critical
$\Rightarrow \exists v \in V(C)$ such that $C - v$ does not have a perfect matching.

By induction hypothesis (Tutte's theorem on $G - v$), $q(C - v - S') > |S'|$ for some $S' \subseteq V(C) \setminus \{v\}$

Parity argument: $|V(C)|$ is odd $\Rightarrow |V(C - v - S')| \equiv |S'| \pmod 2$.
Since $q(H) \equiv v(H) (\operatorname{mod} 2)$, we have $q(C - v - S') \equiv |S'| \pmod 2$.
$\Rightarrow q(C - v - S') \geq |S'| + 2$.

Let $T \coloneqq S \cup \{v\} \cup S'$.
$q(G - T) = q(G - S) - 1 + q(C - v - S') \geq q(G - S) - 1 + |S'| + 2$.

$\Rightarrow r(T) = q(G - T) - (|S| + 1 + |S'|) \geq q(G - S) + |S'| + 1 - |S| - 1 - |S'| = r(S)$.

Since $T \supsetneq S$, this contradicts the maximality of $|S|$.

#

Proof of 2 (Matchability):

Suppose S is not matchable with $\mathscr{C}_{G-S}$.

By Hall's theorem on the auxiliary graph H_S, $\exists S' \subseteq S$ such that $|N_{H_S}(S')| < |S'|$.

Consider $T := S \setminus S'$. Any component $C \in \mathscr{C}_{G-S}$ that is **not** a neighbor of S' in H_S is disconnected from S' in G, so it remains a component in $G - T$.

$q(G-T) \geq q(G-S) - |N_{H_S}(S')|$.

$r(T) = q(G-T) - (|S| - |S'|) \geq q(G-S) - |N_{H_S}(S')| - |S| + |S'| = r(S) + \big(|S'| - |N_{H_S}(S')|\big)$.

Since $|S'| > |N_{H_S}(S')|$, we have $r(T) > r(S)$, a contradiction.

#

We have found $S^* = S$ satisfying conditions 1 and 2.

"Moreover $\Longleftarrow$":

Assume $|\mathscr{C}_{G-S^*}| = |S^*|$.
Since S^* is matchable into $\mathscr{C}_{G-S^*}$ and the sizes are equal, there exists a perfect matching M_H in the auxiliary graph H_{S^*}.

Construct a perfect matching for G:
1. For each edge $sC \in M_H$ (where $s \in S^*, C \in \mathscr{C}$), select the edge $sv \in E(G)$ where $v \in V(C)$ (guaranteed by definition of H_S).
2. Since C is critical, $C - v$ has a perfect matching M_C.
3. $M := \{sv \mid sC \in M_H\} \cup \bigcup\limits_{C \in \mathscr{C}} M_C$ is a perfect matching of G.

"Moreover $\Longrightarrow$":
Let G have a perfect matching

By Hall's condition (and the fact S^* is matchable), $|S^*| \leq |\mathscr{C}_{G-S^*}|$.

By Tutte's Theorem (necessary condition), $q(G - S^*) \leq |S^*|$.
Since $\mathscr{C}_{G-S^*}$ consists of odd components, $|\mathscr{C}_{G-S^*}| = q(G - S^*)$.

Therefore, $|\mathscr{C}_{G-S^*}| = |S^*|$. □

Definition 6.22 **l-edge-connected**

A graph G is l-edge-connected if $\forall F \subseteq E(G) : |F| \leq l - 1 \Rightarrow G - F$ is connected

Lemma 6.23

Let G be a 3-regular and 2-edge-connected graph.

Then G has a perfect matching.

Proof: We want to show that G satisfies Tutte's condition: $|S| \geq q(G-S) \quad \forall\ S \subseteq V(G)$.

Let $\mathscr{O} := \{c \in \mathscr{C}_{G-S} \mid v(C) \text{ odd}\}$ be the set of odd connected components in $G - S$.

Let $e_G(S, C) := \#\{e \in E(G) \mid |e \cup S| = 1 = |e \cup C|\}$ be the number of edges between S and a component C.

Consider the edges leaving S. Since G is 3-regular:

$$\sum_{v \in S} d_G(v) = 3|S| = 2e(G[S]) + e_G(S, V(G) \setminus S)$$

Since edges leaving S must go to some component in $G - S$ (either odd or even), we have:

$$3|S| \geq e_G(S, V(G) \setminus S) \geq \sum_{C \in \mathcal{O}} e_G(S, C)$$

Claim: $e_G(S, C) \geq 3 \quad \forall\ C \in \mathcal{O}$

Consider the sum of degrees for vertices in an odd component C:
$\sum_{v \in V(C)} d_G(v) = 3V(C)$

On the other hand, counting edges incident to C:
$\sum_{v \in V(C)} d_G(v) = 2e(C) + e_G(S, C)$

$\Rightarrow 3|V(C)| = 2e(C) + e_G(S, C)$

Since $|V(C)|$ is odd, $3|V(C)|$ is odd. Since $2e(C)$ is always even, $e_G(S, C)$ must be **odd**.

Since G is connected and specifically 2-edge-connected, there must be at least 2 edges leaving any proper subset of vertices
$\Rightarrow e_G(S, C) \geq 2$.

An odd integer greater than or equal to 2 must be at least 3.

$\Rightarrow e_G(S, C) \geq 3$

#

Combining the inequality with the claim:

$$3|S| \geq \sum_{C \in \mathcal{O}} e_G(S, C) \geq \sum_{C \in \mathcal{O}} 3 = 3\ |\mathcal{O}| = 3q(G - S)$$

Dividing by 3 gives $|S| \geq q(G - S)$.

By Tutte's Theorem, G has a perfect matching. □

6.2 The Hopcroft-Karp algorithm

Definition 6.24

A graph G is a **$(v_1, \ldots, v_r)$-rooted layered graph** if $\exists$ disjoint $N_0, \ldots, N_h \subseteq V(G) : V(G) = N_0 \cup \ldots \cup N_h$ and $N_0 = \{v_1, \ldots, v_r\}$ and $E(G) \subseteq \bigcup_{i \in [\![h]\!]} E(K[N_{i-1}, N_i])$.

We call $N_0, \ldots, N_h$ are called the **layers** of G.

We define $L(G) := N_h$ (the last layer of G).

Definition 6.25

Given a layered graph G (with layers $N_0, \ldots, N_h$) and $M \subseteq G$, G is called **M-alternating** if $\forall i \in \{0, 1, \ldots, h-1\} : E(G[N_i, N_{i+1}]) \subseteq \begin{cases} E(G) \setminus E(M) & \text{if } i \equiv 0 \pmod 2 \\ E(M) & \text{if } i \equiv 1 \pmod 2 \end{cases}$.

(with $G[N_i, N_{i+1}]$ being the bipartite subgraph of G induced by the vertex sets N_i and N_{i+1} from definition above)

Algorithm 6.26 **The Hopcroft-Karp algorithm**

Input: a bipartite graph $G[A, B]$

Set $M_0 := \emptyset$ and $i := 0$
While M_i is not a maximum matching, do:

1. Set $s := |A \setminus V(M_i)|$ and label the vertices in S as $\{v_1, ..., v_s\}$
2. Use BFS to build a $(v_1, ..., v_s)$-rooted layered M_i-alternating graph T_i. Stop building T_i when $L(T_i) \setminus V(M_i) \neq \emptyset$ or no new vertices can be added.
3. Find a maximal collection $\mathcal{P}_i$ of vertex-disjoint shortest M_i-augmenting paths in T_i
4. Set $M_{i+1} = M_i \Delta P^{(1)} \Delta P^{(2)} \Delta ... \Delta P^{(l)}$ (with $P^{(1)}, ..., P^{(l)} \in \mathcal{P}_i$)
5. $i \leftarrow i + 1$

Output: a maximal matching M_i

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Lemma 6.27

Let G be a bipartite graph and $M \subseteq G$ a matching.

Then, M is maximal $\Longleftrightarrow$ G has no M-augmenting path.

Proof: "$\Longrightarrow$": Note 6.9

"$\Longleftarrow$": exercise $\square$

Lemma 6.28

Let G be a bipartite graph and $M_i \subseteq G$ be a matching obtained in the i-th iteration of the Hopcroft-Karp algorithm.

Let P a shortest M_i-augmenting path in G.

Then, $P \subseteq T_i$ (where T_i is the layered graph constructed in the i-th iteration of the Hopcroft-Karp algorithm).

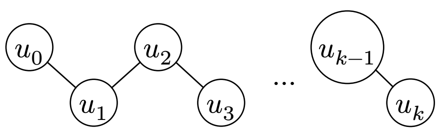
Corollary 6.29

The Hopcroft-Karp algorithm works correctly (i.e., outputs a maximum matching).

Proof: By Lemma 6.28, all shortest M_i-augmenting paths are contained in T_i. $\Rightarrow |M_i| < |M_{i+1}| \leq \frac{v(G)}{2}$ until $\nexists M_i$-augmenting path.

By Lemma 6.27, when no M_i-augmenting path exists, M_i is maximal. $\square$

Proof of Lemma 6.28: As $P = u_0 u_1 ... u_k$ is M_i-augmenting, P is also M_i-alternating and $u_0 \in A \setminus V(M_i)$ and $u_k \in B \setminus V(M_i)$:



Let N_j be the j-th layer of T_i.
Let $N_{\leq j} := N_0 \cup \ldots \cup N_j$

Claim: $\forall j' \in \{0, \ldots, k\} : u_{j'} \in N_{\leq j'}$

Proof by induction on j':
Base case: $j' = 0$, then $u_0 \in N_0 = S = A \setminus V(M_i)$

- $u_{j'} \in N_{\leq j'}$ and $u_{j'}u_{j'+1} \in E(M_i) \Rightarrow u_{j'+1} \in N_{\leq j'+1}$
  because in T_i, odd layers (B-side) expand only along matched edges to the next even layer, so BFS discovers $u_{\{j'+1\}}$ by level $j' + 1$.
- $u_{j'} \in N_{\leq j'}$ and $u_{j'}u_{j'+1} \in E(G) \setminus E(M_i) \Rightarrow u_{j'+1} \in N_{\leq j'+1}$
  because in T_i, even layers (A-side) expand only along unmatched edges to the next odd layer, so BFS discovers $u_{\{j'+1\}}$ by level $j' + 1$.

#

By the claim, $V(P) \subseteq V(T_i)$ and $u_k \in L(T_i)$ (since $u_k \in B \setminus V(M_i)$ by the definition of T_i in the algorithm).

In particular, the height of T_i is k.

Goal: $u_j \in N_j \forall j$ (as this implies $u_{j-1}u_j \in T_i[N_{j-1}, N_j]$)

Define $f : \{0, \ldots, k\} \to \{0, \ldots, k\}$ as $f(j) = i \iff u_j \in N_i$. (i.e., $f(j)$ is the layer index of vertex u_j in T_i)

Observation: $f(j) > f(j-1) \Rightarrow |f(j) - f(j-1)| \leq 1$ because BFS levels are alternating distances:
A shortest alternating path to u_{j-1} followed by $u_{j-1}u_j$ has length $f(j-1) + 1$, so $f(j) \leq f(j-1) + 1$.

Since $f(0) = 0$ and $u_k \in L(T_i)$ gives $f(k) = k$, we get

$$k = f(k) - f(0) = \sum_{j=1}^{k} \underbrace{f(j) - f(j-1)}_{\leq 1}.$$

The sum has k terms, each ≤ 1, yet totals k, hence every term equals 1. Therefore $f(j) = j$ for all j, i.e., $u_j \in N_j$ and $u_{j-1}u_j \in T_i[N_{j-1}, N_j]$. □

> **Lemma 6.30**
>
> Let G be a bipartite graph, $M \subseteq G$ a matching and $P \subseteq G$ a shortest M-augmenting path in G.
>
> If Q is a $M\Delta P$-augmenting path in G, then $e(Q) \geq e(P) + e(P \cap Q)$

Proof: P is M-augmenting $\Rightarrow |M\Delta P| = |M| + 1$
Q is $(M\Delta P)$-augmenting $\Rightarrow |N| = |M| + 2$, with $N := M\Delta P\Delta Q$.

Consider $H := M\Delta N = P\Delta Q$.
H consists of vertex-disjoint paths and cycles (max degree 2).
Since $|N| = |M| + 2$, H must contain at least two components where N-edges exceed M-edges.
Thus, there exist two vertex-disjoint M-augmenting paths P_1, P_2 in H.

Since P is a shortest M-augmenting path, $e(P_1) \geq e(P)$ and $e(P_2) \geq e(P)$.

Using $e(A\Delta B) = (e(A) - e(A \cap B)) + (e(B) - e(A \cap B))$:

$$\begin{aligned} e(P) + e(Q) - 2e(P \cap Q) &= e(P \Delta Q) = e(H) \\ &\geq e(P_1) + e(P_2) \geq 2e(P) \end{aligned}$$

$\Rightarrow e(Q) \geq e(P) + 2e(P \cap Q) \geq e(P) + e(P \cap Q)$ □

Lemma 6.31

Let l_i be the length of the shortest M_i-augmenting path in G in the i-th iteration of the Hopcroft-Karp algorithm.

Then, $l_{i+1} \geq l_i + 1$.

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Proof:
Let P be a shortest M_{i+1}-augmenting path in G. Thus $e(P) = l_{i+1}$.
Note that $M_{i+1} = M_i \Delta \left(P^{(1)} \cup \dots \cup P^{(l)}\right)$.

Case 1: $V(P) \cap V\left(P^{(j)}\right) = \emptyset \forall j \in \{1, \dots, l\}$

Since P is vertex-disjoint from all $P^{(j)}$, it is not affected by the augmentation. Thus, P is also an M_i-augmenting path.

Since $\left\{P^{(1)}, \dots, P^{(l)}\right\}$ is a **maximal** set of vertex-disjoint shortest M_i-augmenting paths, and P is disjoint from all of them, P cannot be a shortest path (otherwise it would have been included in the set).

$\Rightarrow e(P) > l_i \Rightarrow e(P) \geq l_i + 1$.

Case 2: $\exists j \in \{1, \dots, l\} : V(P) \cap V\left(P^{(j)}\right) \neq \emptyset$

Let $k := \max\left\{j : V(P) \cap V\left(P^{(j)}\right) \neq \emptyset\right\}$

$N := M_i \Delta P^{(1)} \Delta P^{(2)} \Delta \dots \Delta P^{(k-1)}$

Observation 1: $P^{(k)}$ is a shortest N-augmenting path in G.
Because $P^{(1)}, \dots, P^{(k-1)}$ are vertex-disjoint from $P^{(k)}$, augmenting along them does not change the status of edges or vertices in $P^{(k)}$. Thus $e\left(P^{(k)}\right) = l_i$.

Observation 2: P is a $N \Delta P^{(k)}$-augmenting path in G.
Because P is vertex-disjoint from $P^{(k+1)}, \dots, P^{(l)}$ (by the choice of k), the subsequent augmentations do not affect P. Thus, P is augmenting with respect to M_{i+1} if and only if it is augmenting with respect to $N \Delta P^{(k)}$.

We apply Lemma 6.30 with $M = N$ and $Q = P$:

$e(P) \geq e\left(P^{(k)}\right) + e\left(P \cap P^{(k)}\right)$.

Since $V(P) \cap V\left(P^{(k)}\right) \neq \emptyset$, the paths share a vertex. In bipartite matchings, two alternating paths sharing a vertex must share at least one edge (otherwise the alternating parity constraints at the shared vertex would be violated).

$\Rightarrow e\left(P \cap P^{(k)}\right) \geq 1$.

$\Rightarrow e(P) \geq l_i + 1$.

□

Claim

Let M^* be the matching output by the Hopcroft-Karp algorithm on a bipartite graph $G[A, B]$.

Then, $e(P) \geq i \ \forall M_i$-augmenting path $P \subseteq G$ in $M^* \Delta M_i$.

In particular, $\# M_i$-augmenting paths in $M^* \Delta M_i \leq \frac{v(G)}{i}$

Proof: Let P be an M_i-augmenting path in $M^* \Delta M_i$.
Since M^* is a maximum matching, M_i cannot be larger than M^*.

Since P is an M_i-augmenting path, its length must be at least the length of the shortest M_i-augmenting path in G, which is l_i.
From the previous lemma, the shortest path length increases strictly with each phase, so $e(P) \geq l_i \geq i$.

The paths in $M^* \Delta M_i$ are vertex-disjoint.
Let k be the number of M_i-augmenting paths in $M^* \Delta M_i$.
Summing the vertices involved:

$$v(G) \geq \sum_{j=1}^{k} v(P_j) = \sum_{j=1}^{k} (e(P_j) + 1) \geq k(i+1) > ki$$

$\Rightarrow k \leq \frac{v(G)}{i}$ □

Claim

$|M^*| - |M_i| \leq \# M_i$-augmenting paths in $M^* \Delta M_i \leq \frac{v(G)}{2}$

Proof: Consider the symmetric difference $H = M^* \Delta M_i$.
The components of H are isolated vertices, even cycles, or paths.

Since M^* is a maximum matching, H cannot contain any M^*-augmenting paths (otherwise M^* could be improved).
Thus, the only path components in H are:
1. Even length paths (equal number of edges from M^* and M_i).
2. M_i-augmenting paths (more edges from M^* than M_i).

Each M_i-augmenting path contributes exactly 1 to the difference $|M^*| - |M_i|$. Even paths and cycles contribute 0.

$\Rightarrow |M^*| - |M_i| = \# M_i$-augmenting paths in H.

The inequality $|M^*| - |M_i| \leq \#$ paths holds with equality.
The upper bound $\frac{v(G)}{2}$ is trivial as paths are disjoint and consume at least 2 vertices. □

Note **Runtime of Hopcroft-Karp**

Let $n := v(G)$ and $m := e(G)$.
Since each phase (BFS + finding maximal set of paths) takes $O(m)$, we optimize the total runtime by splitting the execution at iteration i:

$$\begin{aligned} T(n,m) &= \underbrace{O(mi)}_{\text{first } i \text{ phases}} + \underbrace{O\left(m \cdot \frac{n}{i}\right)}_{\text{remaining phases}} \\ &= O\left(mi + m\frac{n}{i}\right) \end{aligned}$$

The second term follows from the fact that the shortest path length increases strictly ($l_i \geq i$). By the claim above, after i phases, the remaining number of augmentations needed is at most $\frac{n}{l_i} \leq \frac{n}{i}$.
Since each phase adds at least one edge to M, there are at most $\frac{n}{i}$ phases remaining.

Balancing the terms by choosing $i = \sqrt{n}$:

$$O\left(m\sqrt{n} + m\frac{n}{\sqrt{n}}\right) = O(m\sqrt{n})$$

7 k-Connectivity

Definition 7.1

A graph G is ***k*-connected** if $\forall X \subseteq V(G) : |X| \leq k-1 \Rightarrow G - X$ is connected.

$\kappa(G) := \max\{k \mid G \text{ is } k\text{-connected}\}$ is called the **(vertex) connectivity** of G.

Definition 7.2

For a graph G, $X \subseteq V(G)$ is called a **separator** if $G - X$ is disconnected.

Observation

$\delta(G) \geq \kappa(G)$

Theorem 7.3 **Mader '72**

Let $k \in \mathbb{N}_{\geq 1}$ and G be a graph with $\overline{d}(G) \geq 4k$. Then G contains a $(k+1)$-connected subgraph H with $\overline{d}(H) \geq \overline{d}(G) - 2k + 1$

Proof: Let $d^* = \overline{d}(G)$ and $H \subseteq G$ be so that $e(H) = \min\{e(G') \mid G' \subseteq G, v(G') \geq 2k \text{ and } e(G') > \frac{d^*}{2}(v(G') - k)\}$.

Goal: show that H is $(k+1)$-connected and $\overline{d}(H) > d^* - 2k$.

Claim 1: $v(H) \geq 2k + 1$

$e(H) \geq \frac{d^*}{2}(v(H) - k) \geq 2k \cdot k = 2k^2$

Suppose for contradiction that $v(H) = 2k$.
Then, $e(K_{2k}) = \binom{2k}{2} = k(2k-1) < 2k^2$.

As $H \subseteq K_{2k}$, $e(H) \leq e(K_{2k}) < 2k^2$ is a contradiction.

#

Claim 2: $\delta(H) \geq \frac{d^*}{2} + 1$

Suppose for contradiction that $\exists u \in V(H) : d_H(u) \leq \frac{d^*}{2}$.

By Claim 1, $v(H - u) \geq 2k$.

$$\begin{aligned} e(H-u) &= e(H) - d_H(u) \\ &\geq \frac{d^*}{2}(v(H) - k) - \frac{d^*}{2} \\ &= \frac{d^*}{2}(v(H) - k - 1) \\ &= \frac{d^*}{2}(v(H-u) - k) \end{aligned}$$

This contradicts the choice of H minimizing $e(H)$.

#

Claim 3: $\overline{d}(H) > \overline{d}(G) - 2k$

$\overline{d}(H) = \frac{2e(H)}{v(H)} > d^* \frac{v(H)-k}{v(H)}$

$= d^* - \frac{d^* k}{v(H)}$

$\geq d^* - 2k \qquad$ (by Claim 2, $v(H) \geq \frac{d^*}{2} + 2 > \frac{d^*}{2}$)

Claim 4: H is $(k+1)$-connected

Suppose for contradiction that $\exists X \subseteq V(H) : |X| \leq k$ and $H - X$ is disconnected.

Let $K \in \mathscr{C}_{H-X}$

Let $H_1 \coloneqq H[K \cup X]$
Let $H_2 \coloneqq H - K$

Claim 5: $\forall i \in \{1, 2\} : \exists u \in V(H_i) : N_{H_i}(u) = N_H(u)$

For H_1: take $u \in K$. As X separates K from the rest, $N_H(u) \subseteq K \cap X = V(H_1)$.
For H_2: take $u \in V(H) \setminus (K \cap X)$ (exists as $H - X$ disconnected). As X separates u from K, $N_H(u) \subseteq V(H) \setminus K = V(H_2)$.

By Claim 5, H_i contains a vertex u with $d_{H_i}(u) = d_H(u) \geq \delta(H)$.
By Claim 2, $\delta(H) \geq \frac{d^*}{2} + 1 > 2k$.
Thus, $v(H_i) > d_{H_i}(u) \geq 2k$. Since $H_i \subsetneq H$, by the minimality of H:

$e(H_i) \leq \frac{d^*}{2}(v(H_i) - k) \forall i$

Summing the edges (noting $v(H_1) + v(H_2) = v(H) + |X|$):

$$\begin{aligned} e(H) &\leq e(H_1) + e(H_2) \\ &\leq \frac{d^*}{2}(v(H_1) + v(H_2) - 2k) \\ &= \frac{d^*}{2}(v(H) + |X| - 2k) \end{aligned}$$

$$= \frac{d^*}{2}(v(H) - k) + \frac{d^*}{2}(|X| - k)$$

Since $|X| \leq k$, the term $(|X| - k) \leq 0$.
$\Rightarrow e(H) \leq \frac{d^*}{2}(v(H) - k)$, which contradicts the definition of H.

#

□

Lecture 15 (2025/12/01)

Definition 7.4 **x-y separator**

Let G be a graph, $x, y \in V(G)$ and $U \subseteq V(G) \setminus \{x, y\}$. We say that $\boldsymbol{U}$ **separates** $\boldsymbol{x}$ **and** $\boldsymbol{y}$ or $\boldsymbol{U}$ **is an** $\boldsymbol{x}$**-**$\boldsymbol{y}$ **separator** if every x-y path in G contains a vertex from U.

Definition 7.5

A collcetion $\left\{P^{(1)}, ..., P^{(r)}\right\}$ of x-y paths are (internally) vertex-disjoint if $V\left(P^{(i)}\right) \cap V\left(P^{(j)}\right) = \{x, y\} \forall i \neq j$

Observation

Let $xy \in E\left(\overline{G}\right)$; U is a x-y separator and $\mathcal{P}$ is a collection of vertex disjoint x-y paths. Then, $|U| \geq |\mathcal{P}|$

Theorem 7.6 **Menger's theorem**

Let $xy \in E\left(\overline{G}\right)$. Then,

$$\min\{|U| \mid U \text{ is a } x\text{-}y \text{ separator}\} = \max\{|\mathcal{P}| \mid \mathcal{P} \text{ is a collection of vertex-disjoint } x\text{-}y \text{ paths}\}$$

Proof: "$\geq$": follows directly from the above Observation

#

"$\leq$": by induction on $e(G)$.
Base case: empty graph (trivially true)

Let $xy \in E\left(\overline{G}\right)$ and $k := \min\{|U| \mid U \text{ is a } x\text{-}y \text{ separator}\}$

Goal: show that $\exists$ collection $\mathcal{P}$ of k vertex-disjoint x-y paths in G.
This is obviously true for $k \leq 1$. Assume $k \geq 2$.

Case 1: $\exists w \in N_G(x) \cap N_G(y)$.

Consider $G - w$. Any x-y separator in $G - w$ must have size $\geq k - 1$. (Reason: If $G - w$ had a separator U' with $|U'| < k - 1$, then $U' \cup \{w\}$ would be a separator in G of size $< k$, contradicting minimality).

By induction hypothesis, $\exists k - 1$ vertex-disjoint x-y paths in $G - w$. Adding the path x-w-y yields k paths.

#

Case 2: $\exists x\text{-}y$ separator U in $G : |U| = k, U \nsubseteq N_G(x)$ and $U \nsubseteq N_G(y)$.

> Let C_x be the union of components in $G - U$ not containing y.
> Let C_y be the union of components in $G - U$ not containing x.
>
> Let G_x be the graph obtained from G by contracting C_y into a single vertex y' (and adding edges from y' to all $u \in U$).
>
> Let G_y be the graph obtained from G by contracting C_x into a single vertex x' (and adding edges from x' to all $u \in U$).
>
> Claim: $e(G_x) < e(G)$ and $e(G_y) < e(G)$.
>
> > Since $U \nsubseteq N_G(y)$, the y-side (C_y) is not just a single vertex attached to U; it has internal structure or size that gets removed when we contract it to y'. Thus G_x is strictly smaller than G. The same logic applies to G_y.
>
> Since U is a minimum separator in G, it remains a minimum separator in G_x (separating x and y') and G_y (separating x' and y).
> By IH, G_x has k paths from x to U.
> By IH, G_y has k paths from U to y.
>
> Concatenating these paths at the vertices of U provides k vertex-disjoint paths from x to y in G.

#

Case 3: $N_G(x) \cap N_G(y) = \emptyset$ and $\forall x\text{-}y$ separator U in $G : |U| = k \Rightarrow U \subseteq N_G(x)$ or $U \subseteq N_G(y)$.

> Let P be a shortest x-y path. Let u, v be the vertices in P such that $P = x, u, ..., v, y$.
>
> Observation: $v \notin N_G(x)$ (by minimality) and $v \neq y$.
>
> Let $H = G - uv$ (remove edge uv if it exists, or break path).
>
> - If every x-y separator in H has size $\geq k$, then we're done by IH.
> - Assume $\exists x\text{-}y$ separator W in $H : |W| = k - 1$.
>
> Claim: $W \subseteq N_G(x) \cap N_G(y)$.
>
> > Let $W_u = W \cup \{u\}$ and $W_v = W \cup \{v\}$. Both separate x, y in G and have size k.
> > By the assumption of Case 3:
> > 1. $W_v \nsubseteq N_G(x)$ (as $v \notin N_G(x)$). $\Rightarrow W_v \subseteq N_G(y) \Rightarrow W \subseteq N_G(y)$.
> > 2. $W_u \nsubseteq N_G(y)$ (as $u \in N_G(x)$ and $N(x) \cap N(y) = \emptyset$). $\Rightarrow W_u \subseteq N_G(x) \Rightarrow W \subseteq N_G(x)$.
>
> Thus $W \subseteq N_G(x) \cap N_G(y) = \emptyset$.
> $\Rightarrow |W| = 0 \Rightarrow k = 1$.
> Contradiction (as we assumed $k \geq 2$).

#

□

Corollary 7.7

Let G be a graph.

$$G \text{ is } k\text{-connected} \iff \forall x \neq y \in V(G) \exists k \text{ vertex disjoint } x\text{-}y \text{ paths}$$

Proof: "$\Longleftarrow$": w.l.o.g. $k \geq 1$ (trivial for $k = 0$)

In the worst case, there is the x-y path just consisting of the xy edge, and $k-1$ vertex disjoint x-y paths with at least one vertex.

$\Rightarrow \#$ vertices $\geq k-1+2=k+1$

Let $U \subseteq V(G)$ with $|U| \leq k-1$.
$\Rightarrow v(G-U)=v(G)-|U| \geq (k+1)-(k-1)=2$.

Let $x, y \in V(G) \setminus U$. If $xy \in E(G-U)$, then x and y are connected in $G-u$.

If $xy \notin E(G-U)$, then by Menger's theorem, $\exists x$-y path in $G-U$.

#

"$\Longrightarrow$": Let G be k-connected.
$\Rightarrow G-R$ contains a x-y path $\forall R \subseteq V(G) \setminus \{x, y\}$ with $|R|=k-1$

Case 1: $xy \in E\left(\overline{G}\right)$. Then we're done by Menger's theorem.

Case 2: $xy \in E(G)$. Then $G-xy$ is $k-1$-connected. By Menger's theorem, we can find $k-1$ vertex-disjoint x-y paths in $G-xy$.
The edge xy gives the k-th path. □

7.1 A generalisation of Menger's theorem

Definition 7.8

For sets S, T, we say that a path $P=v_0 \ldots v_k$ is a S-T path if $v_0 \in S$ and $v_k \in T$.

Definition 7.9

We say that $\left\{P^{(1)}, \ldots, P^{(r)}\right\}$ is a collection of (internally) vertex-disjoint S-T paths if
(a) $P^{(i)}$ is a S-T path $\forall i$
(b) $V\left(P^{(i)}\right) \cap V\left(P^{(j)}\right) \subseteq S \cup T \forall i \neq j$

Definition 7.10

U is an S-T separation in G if $G-U$ doesn't have an S-T path and $U \subseteq V(G) \setminus (S \cup T)$

Theorem 7.11

Let G be a graph and $S, T \subseteq V(G)$ with $S \cap T=\emptyset$ and $e(G[S, T])=0$. Then,
$\min\{|U| \mid U$ is S-T separator$\}=$
$\max\{|\mathcal{P}| \mid \mathcal{P}$ collection of vertex-disjoint S-T paths in $G\}$

Proof: Let G' be the graph where S is replaced by a single vertex s and T is replaced by a single vertex t such that $N_{G'}(t)=N_G(T)$ and $N_{G'}(s)=N_G(s)$.

Then, U is a s-t separator in $G' \Longleftrightarrow U$ is a S-T separator in G.

Then, apply Menger's theorem on G'. □

Lecture 16 (2025/12/02)

Theorem 7.12

Let G a graph. If $e(G) \geq 3$ and $\alpha(G) \leq \kappa(G)$, then G has a Hamiltonian cycle. (with $\alpha(G) := \max\{|I| \mid I \subseteq V(G), e(G[I]) = 0\}$)

Proof: Suppose for contradiction that G is not Hamiltonian.
Let C be a longest cycle in G with $V(C) = \{v_1, ..., v_l\}$ (indices mod l).
Since $v(G) > l$, let D be a connected component of $G - C$.
Let $B := N_G(D) \cap V(C) = \{v_i \mid i \in I\}$ be the neighbors of D on C.

Since B separates D from the rest of the graph (or G is connected), $|B| \geq \kappa(G) \geq \alpha(G)$.

Let $S := \{v_{i+1} \mid i \in I\}$ be the successors of the vertices in B along C.
Observation 1: $B \cap S = \emptyset$ (if $v_i, v_{i+1} \in B$, we could extend C through D).
Observation 2: S is an independent set (if $v_{i+1} \sim v_{j+1}$, we could extend C by going $v_i \to$ through $D \to v_j \to v_{j+1} \to ... \to v_{i+1} \to v_{j+1}$).
Observation 3: No vertex in S is adjacent to any vertex in D (otherwise C extends).

Let $x \in V(D)$. Then $A := S \cup \{x\}$ is an independent set.
$|A| = |S| + 1 = |B| + 1 \geq \alpha(G) + 1$.

This contradicts the definition of $\alpha(G)$. □

8 Flows in Digraphs

Definition 8.1 — **Directed graph**

A **directed graph** or **digraph** is a pair (V, E) with $E \subseteq \{(u, v) : u, v \in V, u \neq v\}$

Definition 8.2 — **Capacity function**

Let G be a directed graph. A function $c : E(G) \to \mathbb{R}_{>0}$ is called **capacity function**.

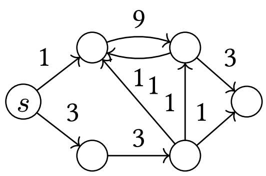
Definition 8.3 — **Flow**

Let G be a digraph, $s \neq t \in V(G)$, $c : E(G) \to \mathbb{R}_{>0}$. We say that $f : V(G) \times V(G) \to \mathbb{R}$ is a (c, s, t)-flow in G if

(1) $0 \leq f(u, v) \leq c(u, v) \quad \forall\ (u, v) \in E(G)$
(2) $f(u, v) = 0 \quad \forall\ (u, v) \notin E(G)$
(3) $\sum_{u \in V(G)} f(v, u) = \sum_{u \in V(G)} f(u, v) \quad \forall\ v \in V(G) \setminus \{s, t\}$ (Kirchhoff current law)

s, t are called the **source** and **sink** of the flow, respectively.

Example



The capacity c is omitted. The flow is marked on edges. Where there are no edges, the flow is 0.

In this definition, s and t are exchangeable. By convention the flow is from s to t.

Definition 8.4 **Value of flow**

Let G be a digraph, $s \neq t \in V(G)$, $c : E(G) \to \mathbb{R}_{>0}$ and f a (c, s, t)-flow in G.

The **value** of the flow f is defined as $|f| = \sum_{u \in V(G)} f(s, u) - \sum_{u \in V(G)} f(u, s)$.

Definition 8.5

The value of a (c, s, t)-flow in G is

$$|f| = \sum_{u \in V(G)} f(s, u) - \sum_{u \in V(G)} f(u, s) = \sum_{u \in V(G)} f(s, u) - f(u, s)$$

Lemma 8.6

Let f be a (c, s, t)-flow in G. Then,

$|f| = \sum_{u \in V(G)} f(u, t) - \sum_{u \in V(G)} f(t, u)$

Proof: $|f| \overset{\text{dfn + 3 (flow dfn)}}{=} \sum_{x \in V(G) \setminus \{t\}} \left(\sum_{u \in V(G)} f(s, u) - f(u, s) \right) = \sum_{u \in V(G)} f(u, t) - f(t, u)$ $\square$

Q/ Given (c, s, t) and a digraph G, what is $\max\{|f| : f \text{ is a } (c, s, t)\text{-flow in } G\}$?

Definition 8.7

A ***s*-*t* cut** of G is a partition $V(G) = S \cup T$ with $s \in S$ and $t \in T$.
For $c : E(G) \to \mathbb{R}_{>0}$ we write $c(S, T) = \sum_{\substack{u \in S \\ v \in T \\ (u,v) \in E(G)}} c(u, v)$. $c(S, T)$ is called the **capacity of the cut**.

Lemma 8.8

Let f be a (c, s, t)-flow in G and let (S, T) be a s-t cut. Then, $|f| \leq c(S, T)$.

Proof:

$$0 = \sum_{u \in S \setminus \{s\}} \left(\sum_{v \in V(G)} f(u, v) - f(v, u) \right) \qquad \text{(Kirchhoff current law)}$$

$$
\begin{aligned}
&= \sum_{u \in S \setminus \{s\}} \left(\sum_{v \in V(G)} f(u,v) - f(v,u) \right) + |f| - |f| \\
&= \sum_{u \in S} \left(\sum_{v \in V(G)} f(u,v) - f(v,u) \right) - |f| \\
&= \sum_{u \in S} \sum_{v \in T} (f(u,v) - f(v,u)) - |f| \\
&= \sum_{\substack{u \in S \\ v \in T}} (f(u,v) - f(v,u)) - |f|
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow |f| &= \sum_{\substack{u \in S \\ v \in T}} (f(u,v) - f(v,u)) \\
&\leq \sum_{\substack{u \in S \\ v \in T}} f(u,v) \\
&\leq \sum_{\substack{u \in S \\ v \in T}} c(u,v) \\
&= c(S,T)
\end{aligned}
$$

□

Note

A (c,s,t)-flow f in G is called **maximum** if $|f| = \max\{|f'| \mid f' \text{ is a } (c,s,t)\text{-flow in } G\}$

A s-t cut (S,T) in G is called **minimum** if $c(S,T) = \min\{c(S',T') \mid (S',T') \text{ is a } s\text{-}t \text{ cut in } G\}$

Theorem 8.9 **Max-Flow Min-Cut Theorem**

Let G be a digraph, $s, t \in V(G)$, $s \neq t$, $c : E(G) \to \mathbb{R}_{>0}$. Then,
$\max\{|f| \mid f \text{ is a } (c,s,t)\text{-flow}\} = \min\{c(S,T) \mid (S,T) \text{ is an } s\text{-}t \text{ cut}\}$

Note

The existance of maximum flows and minimum cuts follows from this analysis theorem:

Let $D \subseteq \mathbb{R}^m$ be a compact and $f : D \to \mathbb{R}$ be a continous function. Then $\exists x \in D : f(x) = \sup_{y \in D} f(y)$

Proof of Theorem 8.9: Let $f : V(G) \times V(G) \to \mathbb{R}$ be a maximal (c,s,t)-flow.

Goal: construct a s-t cut (S,T) with $|f| \geq c(S,T)$. (the other direction follows from the previous lemma)

Let $H = (V(G), \{(u,v) \mid f(u,v) < c(u,v) \text{ or } f(v,u) > 0\})$ a directed graph.
(in other words, $(u,v) \in E(H)$ if we can increase $f(u,v) - f(v,u)$ without violating the first constraint of flow)

Let $S := \{v \in V(H) \mid \exists s \to v \text{ path in } H\}$

Claim: $t \notin S$

Suspect for contradiction that $t \in S$.
Then, there exists an s-t path P in H.

Let $\varepsilon := \min_{(u,v) \in E(P)} \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E(G) \\ f(v,u) & \text{if } (u,v) \notin E(G) \end{cases} > 0$.

We define a new flow f' by updating the flow along edges $e \in E(P)$:

- If $e = (u,v) \in E(G)$: Set $f'(u,v) := f(u,v) + \varepsilon$.
  (Valid since $e \in E(H) \Rightarrow f(u,v) < c(u,v)$ and $\varepsilon \leq c(u,v) - f(u,v)$)
- If $e = (u,v) \notin E(G)$: Then $(v,u) \in E(G)$. Set $f'(v,u) := f(v,u) - \varepsilon$.
  (Valid since $e \in E(H) \Rightarrow f(v,u) > 0$ and $\varepsilon \leq f(v,u)$)

For all edges not in P, $f' = f$.

Then $|f'| = |f| + \varepsilon > |f|$, which contradicts the maximality of f.

$\Rightarrow (S, V \setminus S)$ is a s-t cut in G.

Let $T := V(G) \setminus S$. Consider any $u in S$ and $v in T$. By definition of S, there is no edge from S to T in H (otherwise v would be reachable from s). $\Rightarrow (u,v) \notin E(H)$.

By the definition of $E(H)$, $(u,v) \notin E(H)$ implies two things:
1. It is not true that $f(u,v) < c(u,v) \Rightarrow f(u,v) = c(u,v)$.
2. It is not true that $f(v,u) > 0 \Rightarrow f(v,u) = 0$.

We calculate the value of the flow across the cut:

$$\begin{aligned} |f| &= \sum_{\substack{u \in S \\ v \in T}} (f(u,v) - f(v,u)) \\ &= \sum_{\substack{u \in S \\ v \in T}} (c(u,v) - 0) \\ &= c(S,T) \end{aligned}$$

□

Lecture 17 (2025/12/08)

Theorem 8.10

Let G be a digraph, $s, t \in V(G)$, $s \neq t$, $c : E(G) \to \mathbb{N}_{>0}$. Then,

$$\exists \text{ maximum } (c,s,t)\text{-flow } f : V(G) \times V(G) \to \mathbb{R}_{\geq 0} \text{ with } f(x,y) \in \mathbb{N}_{\geq 0} \quad \forall\ x, y \in V(G)$$

.

Proof:
Recall: $\min\{c(S,T) : (S,T) \text{ is an } s\text{-}t \text{ cut}\} \geq |f| = \sum_{v \in V(G)} (f(s,v) - f(v,s))$ for any (c,s,t)-flow f.

Goal: Find a (c,s,t)-flow $f : V(G) \times V(G) \to \mathbb{N}_{\geq 0}$ and an s-t cut (S,T) with $c(S,T) \leq |f|$.

Let $i \geq 0$ and $f_i : V(G) \times V(G) \to \mathbb{N}_{\geq 0}$ be a flow (not necessarily maximum).
Let $H_i = (V(G), \{(u,v) \mid f_i(u,v) < c(u,v) \text{ or } f_i(v,u) > 0\})$.
Let $S_i = \{v \in V(H_i) \mid \exists s \to v \text{ path in } H_i\}$

Case 1: $t \in S_i$

> Since $t \in S_i$, there exists an s-t path P in H_i.
> We define f_{i+1} by updating the flow along edges $e \in E(P)$:
>
> - If $e = (u, v) \in E(G)$: Set $f_{i+1}(u, v) := f_i(u, v) + 1$.
>   (Valid since $e \in E(H_i) \Rightarrow f_i(u, v) < c(u, v)$)
> - If $e = (u, v) \notin E(G)$: Then $(v, u) \in E(G)$. Set $f_{i+1}(v, u) := f_i(v, u) - 1$.
>   (Valid since $e \in E(H_i) \Rightarrow f_i(v, u) > 0$)
>
> For all edges not in P, $f_{i+1} = f_i$.
>
> **Result:** Flow conservation is preserved at internal nodes, while the net flow out of s increases.
>
> $$|f_{i+1}| = |f_i| + 1$$
>
> We repeat this step. Since $|f|$ increases by integers and is bounded by the total capacity, this process terminates (transitioning to Case 2).

Case 2: $t \notin S_i$

> Set $f := f_i$, $S = S_i$ and $T = V(G) \setminus S$.
> Then, (S, T) is a s-t cut in G.
>
> $$\begin{aligned} 0 &= \sum_{\substack{u \in S \setminus \{s\} \\ v \in V(G)}} (f(u, v) - f(v, u)) + |f| - |f| \\ &= \sum_{\substack{u \in S \\ v \in V(G)}} (f(u, v) - f(v, u)) - |f| \\ \Rightarrow |f| &= \sum_{\substack{u \in S \\ v \in V(G)}} (f(u, v) - f(v, u)) = \sum_{\substack{u \in S \\ v \in T}} (f(u, v) - f(v, u)) \\ &= \sum_{\substack{u \in S \\ v \in T}} (f(u, v) - 0) \\ &= c(S, T) \end{aligned}$$

□

> **Note**
>
> The above construction can be done algorithmically. The algorithm for that is called Ford-Fulkerson algorithm.

9 Planar graphs

Definition 9.1

A graph G is called **planar** if $\exists$ injective map $p : V(G) \to \mathbb{R}^2$ and continous maps $h_{uv} : [0,1] \to \mathbb{R}^2 \quad \forall uv \in E(G)$ so that:

(1) $\{h_{uv}(0), h_{uv}(1)\} = \{p(u), p(v)\} \forall uv \in E(G)$
(2) $\forall uv, xy \in E(G), uv \neq xy$ we have

$$h_{xy}([0,1]) \cap h_{uv}([0,1]) = \{p(x), p(y)\} \cap \{p(u), p(v)\}$$

Set $h = \left\{h_{xy} \mid xy \in E(G)\right\}$. We refer to (p, h) as a **plane embedding of G**.

(G, p, h) is called a **plane graph**.

$h(G) := \cup_{xy \in E(G)} h_{xy}([0,1])$ is called a **drawing of G**

Observation

$h(G)$ is compact.

Theorem 9.2 **Fáry's theorem**

A graph G is planar if and only if $\exists$ a plane embedding (p, h) of G so that $h_{uv}([0,1])$ is a straight line segment $\forall uv \in E(G)$ (i.e. $h_{uv}(t) = p(u) + t(p(v) - p(u))$)

Proof: Without proof $\square$

Definition 9.3

Let G be a planar graph and (p, h) be a plane embedding of G. For $x \in \mathbb{R}^2$, define

$$F^{(x)} := \{y \in \mathbb{R}^2 \mid \exists g : [0,1] \to \mathbb{R}^2 \text{ continuous with } g([0,1]) \subseteq \mathbb{R}^2 \setminus h(G),$$
$$g(0) = x \text{ and } g(1) = y\}$$

F is called a **face of (G,p,h)** if $\exists x \in \mathbb{R}^2$ with $F = F^{(x)}$.

$\mathcal{F}(G) := \{F \mid F \text{ is a face of } (G, p, h)\}$ is the set of all faces of (G, p, h).

We denote with $f(G) := |\mathcal{F}(G)|$ the number of faces of (G, p, h).

Observation

(a) Each face is open in $\mathbb{R}^2$.

Proof: Let F be a face and $x \in F$. By definition of F, there exists a continuous function $g : [0,1] \to \mathbb{R}^2$ with $g([0,1]) \subseteq \mathbb{R}^2$ without $h(G)$, $g(0) = x$ and $g(1) = y$ for some $y \in \mathbb{R}^2$.

Since $\mathbb{R}^2 \setminus h(G)$ is open (as $h(G)$ is compact), there exists an open ball $B_\varepsilon(x) \subseteq \mathbb{R}^2$ without $h(G)$.

Let $z \in B_\varepsilon(x)$. Define the continuous function $g' : [0,1] \to \mathbb{R}^2$ by

$$g'(t) = \begin{cases} g(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(1) + 2\left(t - \frac{1}{2}\right)(z - g(1)) & \text{if } \frac{1}{2} < t \leq 1 \end{cases}$$

Then, $g'([0,1]) \subseteq \mathbb{R}^2$ without $h(G)$, $g'(0) = z$ and $g'(1) = y$.
Thus, $z \in F$. Since $z \in B_\varepsilon(x)$ was arbitrary,
$B_\varepsilon(x) \subseteq F$. Thus, F is open. □

(b) As $h(G)$ is compact, G has only one unbounded face.
We call this face **outer face** and the other ones **inner face**.

Proof: As $h(G)$ is compact, there exists a sufficiently large ball $B_R(0)$ so that $h(G) \subseteq B_R(0)$.

Let $x, y \in \mathbb{R}^2$ with $\|x\|, \|y\| > R$. We can find a continous function $g : [0,1] \to \mathbb{R}^2$ with $g([0,1]) \subseteq \mathbb{R}^2$ without $B_R(0)$, $g(0) = x$ and $g(1) = y$ (e.g. a circular arc outside $B_R(0)$).

Then, $g([0,1]) \subseteq \mathbb{R}^2 \setminus h(G)$, $g(0) = x$ and $g(1) = y$.
Thus, $x \in F^{(y)}$. Since x, y were arbitrary, there is only one unbounded face. □

Definition 9.4

Let G be a planar graph. $X \subseteq \mathbb{R}^2$ is called **path-connceted** if $\forall p, q \in X$:
$\exists$ continous map $\varphi : [0,1] \to X$ such that $\varphi(0) = p$ and $\varphi(1) = q$.

Definition 9.5

An open set $\mathcal{U} \subseteq \mathbb{R}^2$ is connected if $\forall$ open sets $\sigma_1, \sigma_2 \subseteq \mathcal{U}$; $\sigma_1, \sigma_2 \neq \emptyset$; $\sigma_1 \cap \sigma_2 = \emptyset$; we have $\mathcal{U} \neq \sigma_1 \cup \sigma_2$.

Theorem 9.6

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be open. Then, $\mathcal{U}$ is connected if and only if $\mathcal{U}$ is path-connected.

Proof: Without proof □

Note

By above observation and above theorem, every face is a connected open set.

Definition 9.7

$p \in \mathbb{R}^2$ is called **accessible from $\mathcal{U}$** if $\exists$ continous map $\varphi : [0,1] \to \mathbb{R}^2$ with $\varphi([0,1)) \subseteq \mathcal{U}$ and $\varphi(1) = p$.

Definition 9.8

Let $\varphi : [0,1] \to \mathbb{R}^2$ be a continous map so that $\varphi(0) = \varphi(1)$ and $\varphi|_{[0,1)}$ is injective. We say that φ is a **Jordancurve**.

Definition 9.9

The frontier (or boundary) of a set $S \subseteq \mathbb{R}^2$ is

$$\partial S := \{x \in \mathbb{R}^2 \mid \forall r > 0 : B(x,r) \cap S \neq \emptyset \text{ and } B(x,r) \cap (\mathbb{R}^2 \setminus S) \neq \emptyset\}$$

Theorem 9.10 **Jordan's Curve Theorem**

Let $C : [0,1] \to \mathbb{R}^2$ be a Jordancurve. Then, $\exists$ open $\mathcal{U}_1, \mathcal{U}_2$ with $\mathbb{R}^2 \setminus C([0,1]) = \mathcal{U}_1 \cup \mathcal{U}_2$ so that

1. $U_1 \cap U_2 = \emptyset$
2. U_1 and U_2 are connected
3. $\partial \mathcal{U}_1 = \partial \mathcal{U}_2 = C([0,1])$
4. $p \in C([0,1])$ is accessible from U_1 and U_2.

Proof: Without proof □

Corollary 9.11

Let G be a planar graph and (p,h) be a plane embedding of G.
Then, $\forall F \in \mathcal{F}(G) : \partial F = \{p \in \partial F : p \text{ is accessible from } F\}$

Lemma 9.12

Let G be a planar graph and (p,h) be a plane embedding of G and $F \in \mathcal{F}(G)$.
Then, $\exists H \subseteq G : \partial F = \cup_{xy \in E(H)} h_{xy}([0,1])$.

Proof: Without proof □

Notation 9.13

We denote by $G[F]$ the graph H from above lemma.

Lemma 9.14

Let G be a planar graph and (p,h) be a plane embedding of G. Then,

$$G \text{ is a forest} \iff |\mathcal{F}(G)| = 1$$

Proof: Follows from Theorem 9.10, Corollary 9.11 and Lemma 9.12. □

Lemma 9.15

Let G be a planar graph and (p, h) be a plane embedding of G. Let $e \in E(G)$. Then, $|\{F \in \mathscr{F}(G) \mid e \in E(G[F])\}| = \begin{cases} 1 & \text{if } \nexists \text{ cycle } C \text{ in } G \text{ with } e \in E(C) \\ 2 & \text{otherwise} \end{cases}$

Proof: Without proof □

Lemma 9.16

Let G be a 2-connected planar graph. Then, $\forall F \in \mathscr{F}(G) : G[F]$ is a cycle.

Proof: Without proof □

Lecture 18 (2025/12/09)

10 Euler's Formula and Triangulations

Theorem 10.1 **Euler's formula**

Let (G, h, p) be a connected plane graph. Then, $v(G) - e(G) + f(G) = 2$ (with $f(G) := |\mathscr{F}(G)|$)

Note

The whole theorem is called "Euler's formula" not just the equation.

Proof: Let $n = v(G)$. Induction on $e(G)$.

Basis of induction: Take G to be a tree. Then, $e(G) = n - 1, f(G) = 1$.

Suppose G is not a tree. Let C be a cycle in G and $e \in E(C)$.
$f(G - e) = f(G) - 1$ (because removing the edge connects the exterior face to the interior one)

$v(G - e) - e(G - e) + f(G - e) = 2$ (by induction hypothesis)
$\Rightarrow v(G) - e(G) + 1 + f(G) - 1 = 2$ □

Definition 10.2

A plane graph (G, h, p) is **maximal** if $\forall xy \in E\left(\overline{G}\right) : \nexists$ injective continous function $h_{xy} : [0, 1] \to \mathbb{R}^2$ such that $\left(G + xy, h \cup h_{xy}, p\right)$ is a plane graph.

Definition 10.3

A plane graph (G, h, p) is a **plane triangulation** if $G[F] \cong K_3 \quad \forall\ F \in \mathscr{F}(G)$

Theorem 10.4

Let (G, h, p) be a plane graph with $v(G) > 3$. Then,

$$G \text{ is maximal plane} \Leftrightarrow G \text{ is a plane triangulation}$$

Proof:
"$\Longleftarrow$":
Let G be a plane triangulation. Suppose for contradiction that G is not maximal, i.e., we can add an edge $e = xy$ (with $x, y \in V(G), xy \notin E(G)$) such that $G + e$ is still planar. The drawing of e must lie inside some face F of G. Since G is a triangulation, the boundary ∂F forms a K_3. Thus, the endpoints x, y must be vertices of this K_3. However, in a K_3, all vertices are already pairwise adjacent. Thus $xyinE(G)$, which is a contradiction.

"$\Longrightarrow$": Assume G is maximal plane.

We may assume G is connected. (If not, we can pick two vertices u, v in the outer faces of two different connected components and draw an edge between them without crossing any existing edges. This contradicts maximality.)

G is 2-connected:

> Suppose not. Let v be a cut-vertex. Then there exists a face F incident to v whose boundary contains vertices from at least two different components of $G - v$.
>
> Let u and w be neighbors of v on the boundary of F that belong to different components of $G - v$. We can draw an edge uw through the interior of F without crossing existing edges. Since u and w are in different components of $G - v$, they are not adjacent in G.
>
> Thus, $G + uw$ is a plane graph, contradicting the maximality of G.
>
> #

Since G is 2-connected, the boundary of every face F is a cycle.
Suppose there exists a face F with length $|\partial F| \geq 4$. Let v_1, v_2, v_3, v_4 be four consecutive vertices on the boundary of F.

We consider adding a diagonal through F:
- If $v_1 v_3 \notin E(G)$, we can draw the edge $v_1 v_3$ inside F, contradicting maximality.
- If $v_1 v_3 \in E(G)$, then this edge must lie strictly **outside** of F (otherwise F would be divided into two faces). By the Jordan Curve Theorem, the cycle formed by v_1, v_2, v_3 along with the edge $v_1 v_3$ separates v_2 from v_4. Therefore, $v_2 v_4 \notin E(G)$. We can then draw the edge $v_2 v_4$ inside F, contradicting maximality.

Thus, every face must be a K_3, making G a plane triangulation. □

> **Corollary 10.5**
>
> Let (G, h, p) be a plane graph with $v(G) \geq 3$. Then,
>
> $e(G) \leq 3v(G) - 6.$
>
> with equality if and only if G is a plane triangulation.

Proof: Let $n = v(G)$, $m = e(G)$, and $f = f(G)$.
Since G is a simple graph with $n \geq 3$, every face F is bounded by a cycle of length at least 3. Thus, $|\partial F| \geq 3$ for all $F \in \mathscr{F}(G)$.

Summing the boundary lengths of all faces, every edge is counted at most twice (once for each side). Therefore:

$$3f \leq \sum_{F \in \mathscr{F}(G)} |\partial F| \leq 2m$$

$$\Rightarrow f \leq \frac{2m}{3}$$

Using Euler's formula ($n - m + f = 2$), we substitute for f:

$$n - m + \frac{2m}{3} \geq 2$$

Multiplying by 3:

$$3n - 3m + 2m \geq 6$$

$$3n - m \geq 6$$

$$m \leq 3n - 6$$

Equality holds if and only if $f = \frac{2m}{3}$, which happens exactly when $|\partial F| = 3$ for all faces F (and every edge is incident to two faces). By definition, this means G is a plane triangulation. □

Corollary 10.6

K_5 is not planar.

Proof: $e(K_5) = 10 > 9 = 3 \cdot 5 - 6$ □

Lemma 10.7

$K_{3,3}$ is not planar.

Note

We can't use the previous corollary since $e\left(K_{3,3}\right) = 9 < 12 = 3 \cdot 6 - 6$. We can however show a similar corollary:

Corollary 10.8

Let (G, h, p) be a plane graph with $e(G[F]) \geq 4 \forall F \in \mathcal{F}(G)$. Then, $e(G) \leq 2v(G) - 4$

Proof: By assumption, every face F is bounded by at least 4 edges ($e(G[F]) \geq 4$), so $|\partial F| \geq 4$. Summing the boundary lengths over all faces, each edge is counted at most twice (once for each side). Thus:

$$4f(G) \leq \sum_{F \in \mathcal{F}(G)} |\partial F| \leq 2e(G)$$

$\Rightarrow 2f(G) \leq e(G)$

Substituting this into Euler's formula (doubled: $2v(G) - 2e(G) + 2f(G) = 4$):
$2v(G) - 2e(G) + 2f(G) = 4$

$\Rightarrow 2v(G) - 2e(G) + e(G) \geq 4$
$\Rightarrow e(G) \leq 2v(G) - 4$ $\square$

Proof of Lemma 10.7: In $K_{3,3}$ there are no triangles.

We have: $e(G) = 9 > 8 = 2 \cdot 6 - 4 = 2v(G) - 4$. By the corollary above, $K_{3,3}$ can't be planar. $\square$

Definition 10.9

A **polyhedron** $P \subseteq \mathbb{R}^3$ is a bounded set of the form

$$P = \bigcap_{i \in [\![n]\!]} \left\{(x, y, z) \in \mathbb{R}^3 \mid \alpha_{i_1} x + \beta_{i_2} y + \gamma_{i_3} z \leq c_i\right\}$$

We define $P_i := \left\{(x, y, z) \in \mathbb{R}^3 \mid \alpha_{i_1} x + \beta_{i_2} y + \gamma_{i_3} z = c_i\right\}$

We assume n is minimal such that the above representation holds.

An **edge** of P is a non-empty set of the form $P_i \cap P_j \cap P$.

A **vertex** is a non-empty set given by the intersection of 2 edges.

A **face** is the bounded open set (in 2D) in $P_j \setminus \bigcup_{i \neq j} P_i$

Definition 10.10

$P \subseteq \mathbb{R}^3$ is a **platonic solid** if P is a polyhedron and every vertex meets the same number of faces and the boundary of each face is the same regular polygon.

Theorem 10.11

There are exactly 5 platonic solids.

Proof: Let P be a platonic solid and G its associated graph. We view G as a connected planar graph. Let $n = v(G)$, $m = e(G)$, and $f = f(G)$.

Since P is a platonic solid, there exist integers $s, r \geq 3$ such that:
- $s := d_{G(v)}$ for all $v \in V(G)$ (every vertex has degree s).
- $r := |\partial F|$ for all $F \in \mathscr{F}(G)$ (every face is bounded by r edges).

By counting edge incidences:
$ns = 2m \Rightarrow n = \frac{2m}{s}$
$fr = 2m \Rightarrow f = \frac{2m}{r}$

Substituting into Euler's formula ($n - m + f = 2$):
$\frac{2m}{s} - m + \frac{2m}{r} = 2$

Dividing by $2m$:
$\frac{1}{s} - \frac{1}{2} + \frac{1}{r} = \frac{1}{m}$
$\frac{1}{s} + \frac{1}{r} = \frac{1}{2} + \frac{1}{m} > \frac{1}{2}$

We analyze possible integer values for $s, r \geq 3$:
- If $s \geq 4$ and $r \geq 4$, then $\frac{1}{s} + \frac{1}{r} \leq \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, a contradiction.
- Thus, we must have $s = 3$ or $r = 3$.

Case 1 ($s = 3$): $\frac{1}{r} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \Rightarrow r < 6$. Possible values: $r \in \{3, 4, 5\}$.
Case 2 ($r = 3$): $\frac{1}{s} > \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \Rightarrow s < 6$. Possible values: $s \in \{3, 4, 5\}$.

Thus, the possible pairs (s, r) are:
- $(3, 3)$: Tetrahedron
- $(3, 4)$: Cube
- $(3, 5)$: Dodecahedron
- $(4, 3)$: Octahedron
- $(5, 3)$: Icosahedron

For all those, we can manually verify that it's actually a platonic solid. □

Lecture 19 (2025/12/15)

Lemma 10.12

Let G be a plane triangulation with $v(G) \geq 4$. Then, G is 3-connected

Proposition 10.13

Let H be a plane triangulation with $v(H) \geq 4$. Then, $\exists \geq 6$ edges $e \in e(H)$ such that the only triangle's containing e are faces.

Note

We call such edges **good edges**.

Proof: We proceed by induction on $n := v(H)$.
Definition: A triangle is **separating** if both its interior and exterior contain vertices. An edge is **good** if it lies on no separating triangle.

Base case: $n = 4$. $H \cong K_4$. No separating triangles $\Rightarrow$ 6 good edges.

Inductive step: Assume claim holds for $< n$.
Case 1: H has no separating triangles.
All edges are good. $3n - 6 \geq 6$ (since $n \geq 4$).

Case 2: H has a separating triangle T.
T splits H into H_{int} (vertices on/inside T) and H_{ext} (vertices on/outside T). Note $n = v(H_{\text{int}}) + v(H_{\text{ext}}) - 3$.
Edges of T are bad in H. Good edges of H_{int} not in T remain good in H.
In H_{int}, edges of T are good (part of outer face). Thus H inherits (# good edges in H_{int}) $- 3$ edges from H_{int} (same for H_{ext}).

$$\begin{aligned}(\#\text{ good edges in } H) &\geq (\#\text{ good edges in } H_{\text{int}} - 3) + (\#\text{ good edges in } H_{\text{ext}} - 3)\\ &\geq (6-3) + (6-3)\\ &= 6\end{aligned}$$

□

Definition 10.14 **Edge contraction**

Notation: $G/e :=$ the graph obtained from G by contracting e to a new vertex v_e

Formally, $G/e := ((V(G) \setminus e) \cup \{v_e\}, \{f \in E(G) \mid e \cap f = \emptyset\} \cup \{(f \setminus e) \cup \{v_e\} \mid f \in E(G), e \cap f \neq \emptyset\})$, with $v_e \notin V(G)$

Proof of Lemma 10.12:
Observation: e good edge => G/e is a plane triangulation.

> Contracting e merges its endpoints and the two adjacent facial triangles, resulting in the removal of e and the unification of two pairs of boundary edges (removing 2 duplicates).
>
> $e(G/e) = e(G) - 3 = (3v(G) - 6) - 3 = 3(v(G/e)) - 6$
>
> #

Suppose for a contradiction that G is not 3-connected. Then $\exists u, v \in V(G)$ such that $G - u - v$ is disconnected.

Suppose for contradiction that G is not 3-connected. Then $\exists S = \{u, v\} \subseteq V(G)$ that separates G.

Step 1: uv cannot be a good edge.
If uv were a good edge, contracting it would merge S into a single vertex x that separates the graph G/e. But G/e is 3-connected by induction (so it cannot have a cut-vertex).

Step 2: Reduction to K_4.
Since uv is not a good edge (or not an edge at all), there exists a good edge e elsewhere.

> By induction, we can iteratively contract good edges until we reach K_4. Since we never contract u with v (as uv is not good), S maps to a vertex pair S' in K_4 that still separates the graph.

However, K_4 has no separator of size 2. This is a contradiction. □

Theorem 10.15 **Fáry's theorem**

Let G be a maximal planar graph and $K \subseteq G$ so that $K \cong K_3$. Then, G can be drawn in the plane without crossings so that its edges are straight line segments and the outer face F is so that $G[F] = K$

Proof: Let $K \subseteq G$ a triangle and call $V(K\} = \{a, b, c\}$.
Induction on $n = v(G)$.

Base Case: $n = 3$ is trivial ($G = K$).

Inductive Step:
Define the deficiency of a vertex v as $r(v) := 6 - d_{G(v)}$.
Since G is a triangulation, $\delta(G) \geq 3$, so $r(v) \leq 3$ for all v.

Sum of deficiencies:

$$\sum_{v \in V(G)} r(v) = 6n - \sum d_{G(v)} = 6n - 2e(G) = 6n - 2(3n - 6) = 12$$

Since $r(v) \leq 3$, there must be at least $\frac{12}{3} = 4$ vertices with positive deficiency ($d_{G(v)} < 6$).
The outer face has 3 vertices (a, b, c). Thus, there exists an internal vertex z with $d_{G(z)} \leq 5$.

Let $G' := G - z$. The neighbors of z form a cycle C_z of length $d_{G(z)} \leq 5$, bounding a face in G'.
We can add edges to triangulate the interior of C_z (if not already triangulated), resulting in a graph G''.

By induction hypothesis, G'' has a straight-line embedding with outer face K.
Removing the added diagonals yields an embedding of G' where the face bounded by C_z is a polygon with ≤ 5 vertices.

Art gallery theorem:

> In an art gallery shapes as a 5-sided polygon, one needs only one guard to observe the whole gallery.

Thus, we can place z into G' in the interior of the face so that all edges from z to its neighbors can be drawn as straight line segments without crossing any existing edges. □

Note

Fáry's theorem can be used on general planar graphs. This follows from the maximal planar case by adding edges to triangulate faces and then removing them after obtaining a straight-line embedding.

11 Graph Minors and Kuratowski's Theorem

Definition 11.1 **Minor**

Let G be a planar graph. A graph $X \subseteq G$ is a **minor** of G if X can be obtained by a series of edge contractions, edge deletions and vertex deletions.

Lemma 11.2

Let G be a planar graph, $X \subseteq G$ a graph.
X is a minor of $G \iff \exists$ collection of vertex disjoint sets $(V_x)_{x \in V(X)}$ in $V(G)$ such that:
1. $\forall xy \in E(X) : e\left(G\left[V_x, V_y\right]\right) \geq 1$
2. $G[V_x]$ is connected $\forall x \in V(X)$

Proof: Without proof. □

Definition 11.3 **Topological Minor**

S is a **subdivision** of X if S arises from X by replacing edges with pairwise disjoint paths.
If G contains a subdivision of X, then X is a **topological minor**.

Note

Every topological minor is also a minor. (We can contract the paths of the subdivision to retrieve the original edges).

The other implication doesn't generally hold. (Example: A graph with maximum degree 3 can have a K_5 minor, but cannot be a topological minor of K_5 as it lacks vertices of degree 4).

Lecture 20 (2025/12/16)

Theorem 11.4 **Kuratowski**

For every graph G, the following are equivalent:
(i) G is planar
(ii) Neither K_5 nor $K_{3,3}$ is a topological minor of G
(iii) Neither K_5 nor $K_{3,3}$ is a minor of G

Proof: We will not prove this theorem fully. We will only prove (ii) $\Rightarrow$ (i), assuming Theorem 11.7 below. $\square$

Definition 11.5

We say that a graph K is a **Kuratowski graph** if K is a subdivision of $K_{3,3}$ or K_5

Definition 11.6

A **convex embedding** of a graph is a planar embedding in which each face boundary is a convex polygon.

Theorem 11.7 **Main theorem (Tutte '60)**

Let G be a 3-connected graph. If G has no Kuratowski subgraph, then G has a convex embedding in the plane with no 3 vertices in a line.

Proof of (ii) $\Rightarrow$ (i) of Theorem 11.4, assuming Theorem 11.7:
Suppose not. Then, $\exists$ a minimal counterexample: that is, a graph G that is non-planar with no Kuratowski subgraph.

Claim: G is 3-connected.

- G cannot have a 1-separator S (cut-vertex)
  As G was chosen minimally, all components of $G - S$ are planar.
  By Fáry's theorem, all components have straight-line embeddings.
  We can combine these embeddings at S to get a planar embedding of G, contradicting non-planarity.
- G cannot have a 2-separator $S = \{u, v\}$
  As G was chosen minimally, all components of $G - S$ are planar.
  By Fáry's theorem, allx^ components have straight-line embeddings with u and v on the outer face.
  We can combine these embeddings at u and v to get a planar embedding of G, contradicting non-planarity.

#

By Theorem 11.7, G has a convex embedding in the plane with no 3 vertices in a line. Thus, G is planar. $\square$

Lemma 11.8

Let G be a 3-connected graph with $v(G) \geq 5$. Then, $\exists e \in E(G) : G/e$ is 3-connected.

Lemma 11.9

Let G be a graph with no Kuratowski subgraph. Then, G/e has no Kuratowski subgraph for any $e \in E(G)$.

Proof of Lemma 11.9: Suppose for contradiction that it doesn't hold.
Let G be a graph with no Kuratowski subgraph.
Then, $\exists e \in E(G)$ such that G/e has a Kuratowski subgraph K.

Let v_e the contracted vertex in G/e.
Then, v_e has to be a branch vertex (an original vertex of the graph K_5 or $K_{3,3}$ before being subdivided) in the Kuratowski subgraph.

$\Rightarrow$ The Kuratowski subgraph is a subdivision of K_5.

This is a contradiction (draw some images to see this). □

Lecture 21 (2026/01/12)

Proof of Lemma 11.8: Let G be a 3-connected graph with $v(G) \geq 5$.
Suppose for contradiction that for every edge $e \in E(G)$, G/e is not 3-connected.

For $xy \in E(G)$, we call a vertex $z \in V(G)$ such that $G - \{x, y, z\}$ is disconnected a **mate** of xy.

Choose $xy \in E(G)$ and its mate z (exists as G/e is not 3-connected) such that $G - \{x, y, z\}$ has a component H such that $v(H)$ is maximal. Let H' be another component of $G - \{x, y, z\}$ (exists as it's disconnected).

We know that $G - \{x, y\}$ is connected (since G is 3-connected). Thus, $N_G(z) \cap V(H) \neq \emptyset$ (as z connects H to the rest of the graph).

Let $u \in N_G(z) \cap V(H)$. Let $v \in V(G)$ be a mate of zu.

Claim: $G - \{z, u, v\}$ has a component strictly larger than H.

> Let $K = V(H) \cup \{x, y\}$.
>
> **Case 1**: $v \in K$.
> Since G is 3-connected, $\{z, v\}$ cannot separate H from $\{x, y\}$ (a cutset must have size ≥ 3). Thus, $H \setminus \{u, v\}$ remains connected to $\{x, y\} \setminus \{v\}$ in $G - \{z, u, v\}$. Also, H' (another component of $G - \{x, y, z\}$) connects to $\{x, y\} \setminus \{v\}$ (as G is 3-connected). Thus, the component containing H' also contains x, y and almost all of H. Its size is strictly larger than $|H|$ (since $|H'|\ ge 1$).
>
> **Case 2**: $v \notin K$.
> Then v lies in a component $H' \neq H$ of $G - \{x, y, z\}$. Any path from H to v must pass through $\{x, y, z\}$. In $G - \{z, u\}$, paths from $H \setminus \{u\}$ to $\{x, y\}$ do not touch v (as v is separated by $\{x, y\}$). Thus, $(V(H) \setminus \{u\}) \cup \{x, y\}$ is connected in $G - \{z, u, v\}$. Its size is $|H| - 1 + 2 = |H| + 1 > |H|$.

Thus, in both cases we find a component larger than H.
This contradicts the maximality of H. □

Proof of Theorem 11.7: Let G be a 3-connected graph with no Kuratowski subgraph.
Induction on $n = v(G)$.

For $n < 4$, the statement is trivially true.

Base Case: $n = 4$. Then, $G \cong K_4$. We can easily draw K_4 as a convex embedding (start with a triangle and place the fourth vertex inside).

Inductive Step:
By Lemma 11.8, $\exists e \in E(G)$ such that G/e is 3-connected.
By Lemma 11.9, G/e has no Kuratowski subgraph.

Let $H := G/e$

By induction hypothesis, H has a convex embedding with no 3 vertices in a line.
We know $H - v_e$ (with the contracted vertex v_e) is 2-connected. Thus, the face C in $H - v_e$ containing $N_H(v_e)$ is a cycle.

Denote $N_x = N_G(x) \setminus \{y\}$ and $N_y = N_G(y) \setminus \{x\}$.

Recall that $N_H(v_e) = (N_G(x) \cup N_G(y)) \setminus \{x, y\} = N_x \cup N_y$.

Let $\{x_1, ..., x_k\} = N_x$ be labeled clockwise according to C.

As G is 3-connected, $\delta(G) \geq 3$. Thus $k \geq 2$.

Denote by $P_{i,i+1}$ the path in C from x_i to x_{i+1} (indices modulo k).

Case 1: $|N_x \cap N_y| \geq 3$:

> This is not possible, as this would create a Kuratowski subgraph (subdivision of K_5).

Case 2: $|N_x \cap N_y| \leq 2$:

> $|N_x \cup N_y| = d_H(v_e) \geq 3$.
>
> Then there exists $u \in N_y \setminus N_x$. (if $N_y \subseteq N_x$, argue symmetrically by swapping x and y).
>
> Let i be so that $u \in P_{i,i+1}$.
>
> **Case 2A**: $N_y \subseteq V(P_{j,j+1})$ for some j.
>
> > Take te convex embedding of G/e, replace v_e by x. Place y sufficiently close to x to maintain convexity (draw a picture to convince yourself that this works).
>
> **Case 2B**: $N_y \nsubseteq V(P_{j,j+1})$ for any j.
>
> > Let $v \in N_y$ such that $v \notin V(P_{i,i+1})$.
> > Then, G contains a subdivision of $K_{3,3}$ (draw a picture to see this). This is a contradiction to our assumption.

□

12 Coloring Planar Graphs

Q/ How many colors are required to color the regions of maps so that no two adjacent regions receive the same color?

Formally we want to show the maximum of $\chi(G) = \min\{k \mid \exists c : V(G) \to \{1, ..., k\} : c(i) \neq c(j) \forall ij \in E(G)\}$ over all planar graphs.

Definition 12.1

We call $\chi(G) = \min\{k \mid \exists c : V(G) \to \{1, ..., k\} : c(i) \neq c(j) \quad \forall ij \in E(G)\}$ the **chromatic number** of G.

Observation

We need at least 4. Consider K_4.

Lemma 12.2

Let G be a planar graph. Then, $\chi(G) \leq 6$.

Proof: Induction on $n = v(G)$.

Base Case: $n \leq 6$ is trivial.

Inductive Step:
As G is planar, $e(G) \leq 3n - 6$ (see Corollary 10.5). Therefore, $\exists v \in V(G) : d_G(v) \leq 5$ (see also, exercise sheet).
By induction hypothesis, $G - v$ can be colored with 6 colors.
Since v has at most 5 neighbors, there is at least one color left to color v. □

Theorem 12.3 **Heawood' 1890**

Let G be a planar graph. Then, $\chi(G) \leq 5$.

Proof: Induction on $n = v(G)$.

Base Case: $n \leq 5$ is trivial.

Inductive Step:
As G is planar, $e(G) \leq 3n - 6$ (see Corollary 10.5). Therefore, $\exists v \in V(G) : d_G(v) \leq 5$ (see also, exercise sheet).
Set $H = G - v$.
By induction hypothesis, $\chi(H) \leq 5$. Let $c : V(H) \to \{1, ..., 5\}$ with $c(i) \neq c(j) \forall ij \in E(G)$ be a proper coloring of H.

Suppose $c(N_G(v)) = [\![5]\!]$ and hence $d_G(v) = 5$.

For $i \in [\![5]\!]$, let $v_i \in N_G(v)$ with $c(v_i) = i$.

Embed G in the plane and assume $v_1, ...v_5$ are labeled clockwise around v.

Define $H_{1,3} := H[c^{-1}(1) \cup c^{-1}(3)]$ and $H_{2,4} := H[c^{-1}(2) \cup c^{-1}(4)]$

- If $\nexists v_1$-v_3 path in $H_{1,3}$, then we can flip the colors in the component of v_1 in $H_{1,3}$. This frees up color 1 for vertex v.
- Similarly, if $\nexists v_2$-v_4 path in $H_{2,4}$, we can free up color 2 for vertex v.

We may assume, $\exists v_1$-v_3 path in $H_{1,3}$ and $\exists v_2$-v_4 path $\in H_{2,4}$.

Then, these two paths must cross (by Jordan Curve Theorem, see Theorem 9.10), contradicting planarity of G (as they cross but don't share any vertex). □

Theorem 12.4

Let G be a planar graph. Then, $\chi(G) \leq 4$.

Proof: Without proof. □

Lecture 22 (2026/01/13)

13 Graph Coloring

Definition 13.1

A graph G is called ***k*-colorable** if $\chi(G) \leq k$.

Definition 13.2

Let G be a graph. Let $c : V(G) \to [\![k]\!]$.

c is a **proper k-coloring** if $c(i) \neq c(j)\ \forall ij \in E(G)$.

Lemma 13.3

Let G be a graph. Then,

$$\chi(G) \leq \Delta(G) + 1$$

Note

This bound is sharp (the bound is best possible).
Examples where equality holds:

- K_n has $\chi(K_n) = n = \Delta(K_n) + 1$.
- Odd cycles C_{2k+1} have $\chi(C_{2k+1}) = 3 = \Delta(C_{2k+1}) + 1$.

Proof: By induction on $n = v(G)$.

Base Case: $n = 1$ is trivial.

Inductive Step:
Let $v \in V(G)$ with $d_G(v) = \Delta(G)$.
By induction hypothesis, $\chi(G - v) \leq \Delta(G - v) + 1 \leq \Delta(G) + 1$.

We can color $G - v$ with $\Delta(G) + 1$ colors.
Since v has $\Delta(G)$ neighbors, there is at least one color left to color v.

$\Rightarrow \chi(G) \leq \Delta(G) + 1$ □

Theorem 13.4 **Brooks'41**

Let G be a connected graph with $G \neq C_{2k+1}$ and $G \neq K_k \forall k$. Then,

$$\chi(G) \leq \Delta(G)$$

Proof: Let $k := \Delta(G)$. Since $G \neq K_k$ and $G \neq C_{2k+1}$, we have $k \geq 3$.

We distinguish two cases.

Case 1: G is not k-regular.

> There exists a vertex v_n with $d_G(v_n) < k$.
> We order the vertices $v_1, ..., v_n$ such that each v_i $(i < n)$ has a neighbor in $\{v_{i+1}, ..., v_n\}$.
> This can be done by taking a spanning tree rooted at v_n and ordering vertices in reverse order of discovery (leaves first).
>
> We color greedily from v_1 to v_n.
> For $i < n$, v_i has at least one neighbor in $\{v_{i+1}, ..., v_n\}$ (the parent in the tree), so it has at most $d(v_i) - 1 \leq k - 1$ neighbors in $\{v_1, ..., v_{i-1}\}$.
> Thus, there is at least one color free for v_i from the set of k colors.
> For v_n, $d(v_n) < k$, so it has at most $k - 1$ neighbors, leaving a color free.

Case 2: G is k-regular.

Case 2A: G has a cut-vertex x.

> Let $C_1, ..., C_r$ be the connected components of $G - x$. Let $G_i = G[V(C_i) \cup \{x\}]$.
> Since G is connected and $k \geq 3$, the subgraphs G_i are not K_{k+1}.
> By induction (or minimality), $\chi(G_i) \leq k$.
> We can color each G_i with k colors.
> By permuting the names of colors in the subgraphs, we can make the colors agree on x (e.g., set $c(x) = 1$ in all subgraphs).
> This gives a proper k-coloring of G.

Case 2B: G is 2-connected.

> **Claim**
>
> There exist vertices $v_1, v_2, v_n \in V(G)$ such that:
> 1. $G - \{v_1, v_2\}$ is connected
> 2. $v_1 v_2 \notin E(G)$
> 3. $\{v_1, v_2\} \subseteq N(v_n)$
>
> ***Proof of Claim:*** Recall from exercise sheet 9:
> A **block** is a maximal connected subgraph H with no cut vertex in H.
> We showed that a graph G can be decomposed into Blocks $B_1, ..., B_t$ with:
> 1. Every cycle is completely contained in one of the blocks
> 2. $|V(B_i) \cap V(B_j)| \leq 1 \forall i \neq j$
> 3. $E(B_1) \cup ... \cup E(B_t) = E(G)$
>
> Since G is 2-connected, there are no cut-vertices.
> Decompose $G - x$ into blocks.
> In this "blocks tree" (as there are no cycles outside of a block), we have 2 leaves.
>
> Take $v_n = x$.
> Then, x must have a neighbor in each block leaf (otherwise that block would be disconnected from the rest of the graph in G).
> As $G - x$ is not 2-connected, we have at least two blocks.
> Take v_1 and v_2 in different leaves so that $v_1 \nsim v_2$.
>
> Note that $v_1 \neq v_2$ exist, because if $v_1 = v_2$, then v_1 would be a cut-vertex of G (contradicting 2-connectivity).
>
> Since $d_G(x) = k \geq 3$, x has at least 1 neighbor in $G - x - v_1 - v_2$. Therefore $G - \{v_1, v_2\}$ is connected. □
>
> **Using the Claim:** Order the vertices of $G - \{v_1, v_2\}$ as $v_3, ..., v_n$ such that each v_i (for $3 \leq i < n$) has a neighbor with a higher index (similar to Case 1, using a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n).
>
> **Coloring procedure:**
> 1. Assign $c(v_1) = c(v_2) = 1$ (valid since $v_1 v_2 \notin E(G)$).
> 2. Greedily color $v_3, ..., v_{n-1}$. Each v_i has at least one neighbor with higher index (uncolored) and neighbors in $\{v_1, v_2\}$ are fixed to color 1. v_i has neighbors in $\{v_{i+1}, ..., v_n\}$, so it has at most $d(v_i) - 1 = k - 1$ previously colored neighbors. Thus a color is available.

3. For v_n: $N(v_n)$ contains v_1, v_2 (both color 1) and other neighbors in $\{v_3, ..., v_{n-1}\}$. Since v_1 and v_2 share the same color, the number of distinct colors used by $N(v_n)$ is at most $d(v_n) - 1 = k - 1$. Thus, a color is available for v_n.

□

13.1 Bounds on the Chromatic Number

Definition 13.5

Let G be a graph. A **clique** is a subset of vertices $Q \subseteq V(G)$ such that every two distinct vertices in Q are adjacent (i.e., $G[Q]$ is a complete graph).

Definition 13.6

Let G be a graph.
- The **clique number** $\omega(G)$ is the number of vertices in a maximum clique of G.
- The **independence number** $\alpha(G)$ is the number of vertices in a maximum independent set of G.

Observation

$\chi(G) \geq \omega(G)$ as every vertex in the clique is connected to each other and thus needs a different color.

Conjecture 13.7 **Borodin-Kostochka**

Let G be a graph with $\Delta(G) \geq 9$ and $\omega(G) < \Delta(G)$. Then,

$$\chi(G) \leq \Delta(G) - 1$$

Lemma 13.8

Let G be a graph. Then,

$$\chi(G) \geq \frac{v(G)}{\alpha(G)}$$

Proof: Let $k = \chi(G)$ and let $c : V(G) \to [\![k]\!]$ be a proper coloring.
For all $i \in [\![k]\!]$, let $C_i := c^{-1}(i) = \{v \in V(G) : c(v) = i\}$ be the color classes.
Each C_i is an independent set, so $|C_i| \leq \alpha(G)$.
$v(G) = \sum_{i=1}^{k} |C_i| \leq \sum_{i=1}^{k} \alpha(G) = k \cdot \alpha(G) = \chi(G)\alpha(G)$. □

Q/ $\exists f : \mathbb{N} \to \mathbb{N}$ with $\chi(G) \leq f(\omega(G))$? Is the clique number a relevant proxy to the chromatic number? Does a "large" chromatic number force a large clique?

Lemma 13.9

Let H be a graph and $c : V(H) \to [\![\chi(H)]\!]$ be a proper coloring of H. Then,

$$\forall i \in [\![\chi(H)]\!] \ \exists v_i : c(N_H(v_i) \cup \{v_i\}) = [\![\chi(H)]\!]$$

Proof: Suppose for contradiction that the statement is false.
Then $\exists i \in [\![\chi(H)]\!]$ such that $\forall v \in V(H)$, $c(N_H(v) \cup \{v\}) \neq [\![\chi(H)]\!]$.

Consider the set of vertices colored i: $S_i = \{v \in V(H) : c(v) = i\}$.
For any $v \in S_i$, the condition implies that there is some color $j \in [\![\chi(H)]\!]$ that is missing from $c(N_H(v) \cup \{v\})$.
Since $c(v) = i$, this missing color j must be different from i and missing from the neighborhood $N_H(v)$.
We can therefore recolor every $v \in S_i$ to its respective missing color j.
Since S_i is an independent set, recoloring vertices in S_i does not create conflicts with each other, and since we pick colors not present in their neighborhoods, it does not create conflicts with neighbors.
This process eliminates color i entirely, resulting in a proper coloring with $\chi(H) - 1$ colors, which contradicts the definition of $\chi(H)$. □

Theorem 13.10 **Mycielski**

For every $k \in \mathbb{N}$, there exists a triangle-free graph G_k such that $\chi(G_k) \geq k$.

Proof: By induction on k. **Base case:** $k = 3$. Take $G_3 = C_5$ (5-cycle). C_5 is triangle-free and $\chi(C_5) = 3$.

Inductive step: Given a triangle-free graph G_k with $\chi(G_k) \geq k$, we construct G_{k+1} as follows:
Let $V(G_k) = \{v_1, ..., v_n\}$.
Create new vertices $W_k = \{v'_1, ..., v'_n\}$ (copies of $V(G_k)$).
Create a vertex z.
$V(G_{k+1}) = V(G_k) \cup W_k \cup \{z\}$.

Edges of G_{k+1}:
- All edges of G_k.
- For each $v_i \in V(G_k)$, connect v'_i to all neighbors of v_i in G_k. ($N_{G_{k+1}}(v'_i) = N_{G_k}(v_i) \cup \{z\}$)
- Connect z to all vertices in W_k.
- W_k is an independent set.

1. G_{k+1} is triangle-free:
- No triangles in G_k by assumption.
- z is only connected to W_k (independent), so no triangle involves z.
- If a triangle involves $v'_i \in W_k$, it must be v'_i, x, y where $x, y \in V(G_k)$. But v'_i connects to neighbors of v_i. So $x, y \in N_{G_k}(v_i)$. This implies x, y are neighbors in G_k, so v_i, x, y would be a triangle in G_k, which is a contradiction.

2. $\chi(G_{k+1}) \geq k + 1$: Suppose for contradiction $\chi(G_{k+1}) = k$. Let $c : V(G_{k+1}) \to [k]$ be a proper coloring.
Assume w.l.o.g. $c(z) = k$.
Then $W_k \subseteq V(G_{k+1})$ must be colored with $\{1, ..., k - 1\}$ (since z is connected to all of W_k).

We try to define a coloring c' on G_k using $k - 1$ colors.
Let $c'(v_i) = c(v_i)$ if $c(v_i) \neq k$.
If $c(v_i) = k$, we set $c'(v_i) = c(v'_i)$. (Note $c(v'_i) \neq k$ because $v'_i \in W_k$ and $c(z) = k$).

Check properness of c' on G_k:
Let $v_i v_j \in E(G_k)$.
- If $c(v_i) \neq k$ and $c(v_j) \neq k$, then $c'(v_i) = c(v_i) \neq c(v_j) = c'(v_j)$.
- If $c(v_i) = k$, then $c(v_j) \neq k$ (proper in G_{k+1}). $c'(v_i) = c(v'_i)$ and $c'(v_j) = c(v_j)$. Since v'_i connects to $N_{G_k}(v_i)$ in G_{k+1}, $v'_i v_j \in E(G_{k+1})$. Thus $c(v'_i) \neq c(v_j)$, so $c'(v_i) \neq c'(v_j)$.

- Both k is impossible.

Thus c' is a $(k-1)$-coloring of G_k, contradicting $\chi(G_k) \geq k$. □

Lecture 23 (2026/01/19)

Theorem 13.11 **Erdős '59**

For every $k, g \in \mathbb{N}$, there exists a graph G with $g(G) > g$ and $\chi(G) > k$.

14 Probabilistic Method

Definition 14.1 **Erdös-Rényi random graph (binomial random graph)**

For $n \in \mathbb{N}$ and $p \in [0,1]$, the **Erdős-Rényi random graph model** (or binomial random graph) is defined by the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_{n,p})$ where:

- Ω_n is the set of all graphs with vertex set $\{1, \ldots, n\} = [\![n]\!]$.
  $\Rightarrow |\Omega_n| = 2^{\binom{n}{2}}$
- $\mathcal{F}_n = P(\Omega_n) = \{A \subseteq \Omega_n\}$
  $\Rightarrow |\mathcal{F}_n| = 2^{|\Omega_n|} = 2^{2^{\binom{n}{2}}}$
- $\mathbb{P}_{n,p} : \mathcal{F}_n \to [0,1]$ is the probability measure defined for any single graph $G \in \Omega_n$ as:
  $\mathbb{P}_{n,p}(\{G\}) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$

The **random graph** $G(n,p)$ is the identity random variable $G(n,p) : \Omega_n \to \Omega_n$ defined by $\omega \mapsto \omega$.

Intuition: We sample $G(n,p)$ by including every possible edge $e \in \binom{[\![n]\!]}{2} = \{A \subseteq [\![n]\!] : |A| = 2\}$ independently with probability p.

Definition 14.2 **Probabilistic Terminology**

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the probability space defined above.

- An **event** is a subset $A \subseteq \Omega$ (i.e., $A \in \mathcal{F}$).
  The probability of an event A is $\mathbb{P}(A) = \sum_{G \in A} \mathbb{P}(\{G\})$.
  Consequently, for any set $S \subseteq \Omega_n$:
  $\mathbb{P}(G(n,p) \in S) = \sum\limits_{H \in S} \mathbb{P}(G(n,p) = H)$
  (since the events $\{G(n,p) = H\}$ for $H \in S$ are disjoint).
- A **random variable** is a function $X : \Omega \to \mathcal{S}$ for some set $\mathcal{S}$.
  (Commonly $\mathcal{S} = \mathbb{R}$, e.g., $X(G) = e(G)$).
- If $\mathcal{S} \subseteq \mathbb{R}$, the **expectation** of X is defined as:

$$\mathbb{E}(X) := \sum_{G \in \Omega} X(G) \cdot \mathbb{P}(\{G\})$$

Definition 14.3 **Notation**

We often omit the indices n, p from $\Omega, \mathcal{F}, \mathbb{P}$ when clear from context.

For a random variable X and a value a, we use the shorthand:

$$\{X \geq a\} := \{G \in \Omega : X(G) \geq a\}$$

(and analogously for $>, \leq, <, =, \neq, \ldots$).

Therefore, $\mathbb{P}(X \geq a)$ denotes the probability of the event that the random variable X takes a value greater than or equal to a and $\mathbb{P}_{n,p}(G(n,p) = \omega) = \mathbb{P}_{n,p}(\{\omega\})$.

Definition 14.4 **With high probability**

A sequence of events $(A_k)_{k\in\mathbb{N}}$ with $A_k \in \mathcal{F}_n$ holds **with high probability (w.h.p.)** if $\lim_{k\to\infty} \mathbb{P}(A_k) = 1$.

Note **The Probabilistic Method**

If a property $\mathcal{P}$ holds w.h.p. in $G(n,p)$, then $\lim_{n\to\infty} \mathbb{P}(G(n,p) \text{ has } \mathcal{P}) = 1$.

In particular, for a sufficiently large n, $\mathbb{P}(G(n,p) \text{ has } \mathcal{P}) > 0$. Since the probability space is finite, a non-zero probability implies that the set of graphs satisfying $\mathcal{P}$ is non-empty.

Thus, proving that a random graph has a property with probability > 0 (or w.h.p.) implies the existence of a deterministic graph with that property.

Definition 14.5 **indicator random variables**

For $A \in \mathcal{F}_n$, let $\mathbb{1}_A : \Omega_n \to \{0, 1\}$ be given by $\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$.
$\mathbb{1}_A$ is called the **indicator random variable** of event A.

Lemma 14.6

- Linearity of Expectation: For random variables X, Y and scalars $a, b \in \mathbb{R}$:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

- $\mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$ for any event $A \in \mathcal{F}_n$

Proof: Exercise. □

Lemma 14.7 **Markov's inequality**

Let $X : \Omega_n \to \mathbb{R}_{\geq 0}$ be a random variable and $a > 0$. Then,

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

Proof of Lemma 14.7:

$$\begin{aligned} \mathbb{E}(X) &= \mathbb{E}\left(X \cdot \mathbb{1}_{\{X \geq a\}} + X \cdot \mathbb{1}_{\{X < a\}}\right) \\ &\geq \mathbb{E}\left(X \cdot \mathbb{1}_{\{X \geq a\}}\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{E}\Big(a \cdot \mathbb{1}_{\{X \geq a\}}\Big) \\
&= a\mathbb{E}\Big(\mathbb{1}_{\{X \geq a\}}\Big) = a\mathbb{P}(X \geq a)
\end{aligned}
$$

□

Lecture 24 (2026/01/20)

Proof of Theorem 13.11: Fix $k, g \in \mathbb{N}$. We aim to find a graph G' with $\chi(G') > k$ and $g(G') > g$.

Let $\varepsilon := \frac{1}{2g}$ and consider the random graph $G(n, p)$ with $p = n^{\varepsilon - 1}$.
We proceed by the **alteration method** in three steps.

Step 1: Bounding the independence number $\alpha(G)$.
Let $x := \left\lceil \frac{3 \log n}{p} \right\rceil$. We show that $\mathbb{P}(\alpha(G) \geq x) \to 0$.

Let Y_x be the number of independent sets of size x.

$$\mathbb{P}(\alpha(G) \geq x) = \mathbb{P}(Y_x \geq 1) \leq \mathbb{E}(Y_x)$$

We compute the expectation:

$$
\begin{aligned}
\mathbb{E}(Y_x) &= \binom{n}{x}(1-p)^{\binom{x}{2}} \\
&\leq n^x e^{-p\frac{x(x-1)}{2}} && (\text{using } 1 - p \leq e^{-p}) \\
&= \left(ne^{-\frac{p(x-1)}{2}}\right)^x \\
&\leq \left(ne^{-\frac{3}{2}\log n + \frac{p}{2}}\right)^x && \left(\text{using } x \geq \frac{3\log n}{p} \Rightarrow -\frac{px}{2} \leq -\frac{3}{2}\log n\right) \\
&= \left(nn^{-\frac{3}{2}}e^{\frac{p}{2}}\right)^x \\
&= \left(n^{-\frac{1}{2}}(1 + o(1))\right)^x \to 0
\end{aligned}
$$

Thus, **w.h.p.** $\alpha(G) < \frac{3 \log n}{p}$.

Step 2: Bounding the number of short cycles.
Let Z be the number of cycles of length at most g in $G(n, p)$.

Our first guess is bounding $\mathbb{E}(Z)$ w.h.p.:

$$
\begin{aligned}
\mathbb{E}(Z) &= \sum_{j=3}^{g} (\#\text{ cycles of length } j) \cdot p^j \\
&\leq \sum_{j=3}^{g} n^j p^j = \sum_{j=3}^{g} n^j n^{\varepsilon j - j} = \sum_{j=3}^{g} n^{\varepsilon j} \\
&\leq gn^{\varepsilon g} = gn^{\frac{1}{2}} = o(n)
\end{aligned}
$$

Therefore, our attempt to show that $Z = 0$ w.h.p. fails (since $\mathbb{E}(Z) \to \infty$). However, using **Markov's inequality** (Lemma 14.7), we can bound the probability that Z is large:

$$\mathbb{P}\Big(Z \geq \frac{n}{2}\Big) \leq \frac{\mathbb{E}(Z)}{\frac{n}{2}} \leq \frac{2gn^{\frac{1}{2}}}{n} = 2gn^{-\frac{1}{2}} \to 0$$

Thus, **w.h.p.** the number of short cycles is less than $\frac{n}{2}$.

Step 3: Constructing the graph.
Since both events in Step 1 and Step 2 occur w.h.p., for sufficiently large n, there exists a graph G satisfying:
1. $\alpha(G) < \frac{3 \log n}{p}$
2. $\#\{\text{cycles of length} \leq g\} < \frac{n}{2}$

We construct a new graph G' by deleting one vertex from each cycle of length at most g in G.

Properties of G':
- **Girth:** We destroyed all cycles of length $\leq g$, so $g(G') > g$.
- **Size:** We removed at most $\frac{n}{2}$ vertices, so $v(G') \geq n - \frac{n}{2} = \frac{n}{2}$.
- **Independence Number:** $\alpha(G') \leq \alpha(G)$ (deleting vertices cannot increase the size of an independent set).

Finally, we bound the chromatic number of G':

$$\begin{aligned}\chi(G') &\overset{\text{Lemma 13.8}}{\geq} \frac{v(G')}{\alpha(G')} \geq \frac{\frac{n}{2}}{\frac{3 \log n}{p}} = \frac{np}{6 \log n} \\ &= \frac{n^{\varepsilon}}{6 \log n}\end{aligned}$$

Since $\varepsilon > 0$ is fixed, $\frac{n^{\varepsilon}}{6 \log n} \to \infty$ as $n \to \infty$. Therefore, for sufficiently large n, $\chi(G') > k$.

□

Theorem 14.8 **Erdös'62**

$\forall k \exists \varepsilon > 0$ such that $\exists n_0(\varepsilon, k) : \forall n > n_0, \exists$ graph G_k with
- $v(G_k) = n$
- $\chi(G_k) \geq k$
- $\chi\left(G_{k[S]}\right) \leq 4 \ \forall S \subseteq V(G_k) : |S| \leq \varepsilon v(G_k)$

Proof: Let $k \in \mathbb{N}$. We choose constants in the following order:
1. Fix c sufficiently large (depending on k).
2. Fix ε sufficiently small (depending on c).

Let $p := \frac{c}{n}$ and consider the random graph $G(n, p)$. We analyze two events to show that $G := G(n, p)$ satisfies the theorem conditions w.h.p.

Step 1: Global Chromatic Number
We show that $\chi(G) > k$ w.h.p.
Recall that $\{\chi(G) \leq k\} \subseteq \left\{\alpha(G) \geq \frac{n}{k}\right\}$. (This follows from Lemma 13.8.)
Let X be the number of independent sets of size $x := \left\lfloor \frac{n}{k} \right\rfloor$.

$$\begin{aligned}\mathbb{P}(\chi(G) \leq k) &\leq \mathbb{P}(X \geq 1) \leq \mathbb{E}(X) \\ &= \binom{n}{x} (1-p)^{\binom{x}{2}} \\ &\leq \left(\frac{ne}{x}\right)^x e^{-p\frac{x(x-1)}{2}} \\ &\leq \left(ke \cdot e^{-\left(\frac{c}{n}\right) \cdot \left(\frac{n}{2k}\right)\left(\frac{n}{k}-1\right)}\right)^x\end{aligned}$$

$$\leq \left(ke \cdot e^{-\frac{c}{2k}}\right)^x$$

By choosing $c > 2k \ln(ke)$, the base $B = kee^{-\frac{c}{2k}}$ becomes strictly less than 1. Since $x \approx \frac{n}{k}$, as $n \to \infty$, we have $B^x \to 0$. Thus, $\mathbb{E}(X) \to 0$, implying $\chi(G) > k$ w.h.p.

Step 2: Local Chromatic Number
We show that for all $S \subseteq V(G)$ with $|S| \leq \varepsilon n$, $\chi(G[S]) \leq 3$.

Let S be a "bad" set if $|S| \leq \varepsilon n$ and $\chi(G[S]) \geq 4$. If such a set exists, there must be a **minimal** such set S'. A minimal graph with chromatic number ≥ 4 must have minimum degree $\delta \geq 3$. Thus, a minimal bad set S' satisfies $e(G[S']) \geq \frac{3|S'|}{2}$.

Let $\mathcal{A}$ be the collection of sets S with $|S| \leq \varepsilon n$ satisfying this density condition ($e(G[S]) \geq 1.5\ |S|$). It suffices to show $\mathbb{P}(|\mathcal{A}| \geq 1) \to 0$.

We calculate the expected number of such sets, summing over sizes $s = |S|$. Note that since $\delta \geq 3$, we must have $s \geq 4$.

$$\begin{aligned}
\mathbb{E}(|\mathcal{A}|) &= \sum_{s=4}^{\lfloor \varepsilon n \rfloor} \sum_{\substack{S \\ |S|=s}} \mathbb{P}(e(G[S]) \geq 1.5s) \\
&\leq \sum_{s=4}^{\lfloor \varepsilon n \rfloor} \binom{n}{s} \binom{\binom{s}{2}}{1.5s} p^{1.5s}
\end{aligned}$$

Using the bound $\binom{N}{K} \leq \left(\frac{Ne}{K}\right)^K$:

$$\begin{aligned}
\mathbb{E}(|\mathcal{A}|) &\leq \sum_{s=4}^{\varepsilon n} \left(\frac{ne}{s}\right)^s \cdot \left(\frac{s^2 \frac{e}{2}}{1.5s}\right)^{1.5s} \cdot \left(\frac{c}{n}\right)^{1.5s} \\
&= \sum_{s=4}^{\varepsilon n} \left[\frac{ne}{s} \cdot \left(\frac{se}{3}\right)^{1.5} \cdot \left(\frac{c}{n}\right)^{1.5}\right]^s \\
&= \sum_{s=4}^{\varepsilon n} \left[e\left(\frac{ec}{3}\right)^{1.5} \cdot \frac{n}{s} \cdot \left(\frac{s}{n}\right)^{1.5}\right]^s \\
&= \sum_{s=4}^{\varepsilon n} \left[C \cdot \sqrt{\frac{s}{n}}\right]^s
\end{aligned}$$

where $C = e\left(\frac{ec}{3}\right)^{1.5}$ is a constant depending only on c.

For the sum to vanish, we observe two regimes:
- For small s (e.g. $s = 4$), the term is proportional to $\left(n^{-\frac{1}{2}}\right)^4 = n^{-2}$, which goes to 0 as $n \to \infty$.
- For large s, since $\frac{s}{n} \leq \varepsilon$, the base is bounded by $C\sqrt{\varepsilon}$. By choosing ε small enough such that $C\sqrt{\varepsilon} < \frac{1}{2}$, the terms decay geometrically.

Combining these, $\mathbb{E}(|\mathcal{A}|) \to 0$ as $n \to \infty$.

Conclusion (Reflection of $\forall n > n_0$)
We proved that $\mathbb{P}(\chi(G) \leq k) \to 0$ and $\mathbb{P}(\exists \text{ bad local subgraph}) \to 0$. Therefore, $\mathbb{P}(\chi(G) > k \text{ and no bad subgraphs}) \to 1$.

By the definition of a limit, there exists some n_0 such that for all $n > n_0$, this probability is strictly positive (in fact close to 1). This implies the existence of a graph (see Note 14.5) satisfying the theorem statements for all $n > n_0$. □

15 Ramsey Theory

Definition 15.1 **The Party Problem**

Suppose you invite n guests to a party. We want to guarantee that either:
- $\exists$ 3 people who all know each other (mutual acquaintances), or
- $\exists$ 3 people who all do not know each other (mutual strangers).

Question: What is the minimum n required to satisfy this property?

Formally, we ask: What is the minimum n such that for any graph G on n vertices, there is either a triangle (K_3) or an independent set of size 3 (I_3)?

Equivalently, in terms of edge coloring: What is the minimum n which guarantees that for any red-blue coloring of the edges of K_n, there exists a monochromatic triangle?

Claim

The answer to the Party Problem is $n = 6$.

Proof: **Sufficiency ($n = 6$):**
Consider any red-blue coloring of $E(K_6)$. Pick an arbitrary vertex v. $d(v) = 5$. By the Pigeonhole Principle, at least $\lceil \frac{5}{2} \rceil = 3$ edges incident to v share the same color. W.L.O.G., assume v has red edges to neighbors $\{a, b, c\}$.
- Case 1: The edges between $\{a, b, c\}$ are all blue. Then $\{a, b, c\}$ forms a blue K_3.
- Case 2: There exists at least one red edge in $\{a, b, c\}$, say ab. Then $\{v, a, b\}$ forms a red K_3.

In either case, a monochromatic triangle exists.

Necessity ($n = 5$):
We construct a counter-example for K_5:
- Color the outer cycle $(1, 2, 3, 4, 5, 1)$ red.
- Color the inner star (diagonals) blue.

This coloring has no monochromatic triangle since every vertex is incident to exactly two edges of each color. $\square$

Definition 15.2 **Ramsey number**

The **Ramsey number** $R(k, l)$ is the minimum $n \in \mathbb{N}$ such that any red-blue coloring of the edges of K_n contains either a red K_k or a blue K_l.

We define $R(k) := R(k, k)$.

Observations

- From above: $R(3, 3) = 6$.
- $R(2, l) = l$ for all $l \in \mathbb{N}$. (A coloring with all edges blue requires l vertices to form a blue K_l; any red edge yields a red K_2).
- $R(k, l) = R(l, k)$ (Symmetry).

Q/ Is $R(k, l)$ finite for every k, l?

Lemma 15.3 **Recursive Upper Bound**

For all integers $k, l \geq 3$:

$$R(k,l) \leq R(k-1,l) + R(k,l-1)$$

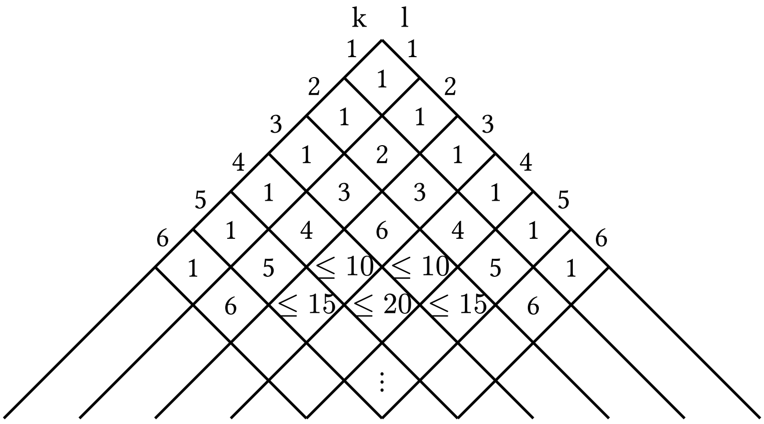
***Proof*:** Let $n = R(k-1,l) + R(k,l-1)$.
Consider any red-blue coloring of edges of K_n.
Pick any vertex v. Let A be the set of vertices connected to v by red edges and B be the set connected by blue edges.
v has $n-1$ edges. By the pigeonhole principle, either $|A| \geq R(k-1,l)$ or $|B| \geq R(k,l-1)$.

- If $|A| \geq R(k-1,l)$:
  Then the subgraph induced by A (or a subset that has size $R(k-1,l)$) contains either a red K_{k-1} or a blue K_l.
  In the first case, adding v gives a red K_k. In the second case, we already have a blue K_l.
- If $|B| \geq R(k,l-1)$:
  Then the subgraph induced by B (or a subset that has size $R(k,l-1)$) contains either a red K_k or a blue K_{l-1}.
  In the first case, we already have a red K_k. In the second case, adding v gives a blue K_l.

In both cases, we find either a red K_k or a blue K_l, proving the lemma. □

Note

This recurrence relation mirrors Pascal's Triangle ($\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$):



Corollary 15.4

$$R(k,l) \leq \binom{k+l-2}{k-1}$$

Specifically for diagonal Ramsey numbers:

$$R(k) \leq \binom{2k-2}{k-1} \leq 2^{2k-2} \leq 4^k$$

***Proof*:** As the numbers in the above diagram are exactly the ones in Pascal's Triangle □

15.1 Bounds on $R(k)$

Q/ How small can $R(k)$ be?

Lemma 15.5 **Constructive Lower Bound**

For all $k \geq 2$:

$$R(k) > (k-1)^2$$

Proof: Construct a graph with $n = (k-1)^2$ vertices. Arrange vertices in a grid (i, j) with $i, j \in \{1, ..., k-1\}$.

- Color edge $((i, j), (i', j'))$ red if $i = i'$ (same row).
- Color edge $((i, j), (i', j'))$ blue if $i \neq i'$ (different rows).

Any red clique must lie within a single row (size at most $k-1$). Any blue clique must have vertices in distinct rows (size at most $k-1$). Thus, no monochromatic K_k exists. □

Theorem 15.6 **Exponential Lower Bound (Erdős '47)**

For all $k \geq 3$:

$$R(k) > 2^{\frac{k}{2}}$$

Proof: Let $n \leq 2^{\frac{k}{2}}$.

Consider a random red-blue coloring of edges of K_n, where each edge is colored red or blue with probability $\frac{1}{2}$ independently.

We want to show: $\mathbb{P}(\{K_n \text{ has no monochromatic } K_k\}) > 0$

Let $A \subseteq G$ with $v(G) = k$. Then,

$$\mathbb{P}(\{A \text{ is monochromatic } K_k\}) = \mathbb{P}(\{A \text{ is red } K_k\}) + \mathbb{P}(\{A \text{ is blue } K_k\}) = 2\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\{K_n \text{ has a monochromatic } K_k\}) &= \binom{n}{k} 2 \left(\frac{1}{2}\right)^{\binom{k}{2}} \\
&= \binom{n}{k} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&= \frac{n!}{(n-k)!k!} 2^{-\binom{k}{2}+1} \\
&= \frac{n!}{(n-k)!k!} 2^{-\frac{k^2}{2}+\frac{k}{2}+1} \\
&\leq \frac{n^k}{k!} 2^{-\frac{k^2}{2}} 2^{\frac{k}{2}} 2 \\
&= \underbrace{\left(n 2^{-\frac{k}{2}}\right)^k}_{\substack{=0 \\ \text{as } n \leq 2^{\frac{k}{2}}}} \frac{2^{\frac{k}{2}} 2}{k!} \\
&\leq \frac{2^{\frac{k}{2}} 2}{k!}
\end{aligned}$$

$$= 2\left(\underbrace{\frac{\sqrt{2}}{1} \cdot \frac{\sqrt{2}}{2}}_{=1} \cdot \underbrace{\frac{\sqrt{3}}{3}}_{<\frac{1}{2}} \cdot \underbrace{\ldots \cdot \frac{\sqrt{2}}{k}}_{<1}\right)$$

$$< 1$$

Thus, $\mathbb{P}(\{K_n \text{ has no monochromatic } K_k\}) > 0$. □

Note — **Current State of Bounds**

We have established:

$$\left(\sqrt{2}\right)^k < R(k) < 4^k$$

Can we improve these bounds?

- **Lower Bound:** Is there an $\varepsilon > 0$ such that $R(k) > \left(\sqrt{2} + \varepsilon\right)^k$?
  Status: Unknown.
- **Upper Bound:** Is there an $\varepsilon > 0$ such that $R(k) < (4 - \varepsilon)^k$?
  Status: Yes. Recent breakthrough (2023) proved for $\varepsilon \approx 10^{-7}$, and very recently (2026) improved to $R(k) < 3.8^k$.

16 Edge Coloring

Definition 16.1 — **proper edge coloring**

An **edge coloring** of a graph G is a function $c : E(G) \to C$ for some set of colors C.
An edge coloring is **proper** if no two adjacent edges share the same color, i.e. $\forall e, e' \in E(G)$ with $e \cap e' \neq \emptyset$: $c(e) \neq c(e')$.

Definition 16.2 — **edge-chromatic number**

The **edge-chromatic number** (or **chromatic index**) $\chi'(G) :=$
$\min\{|C| \mid \exists \text{ proper edge coloring } c : E(G) \to C\}$

Note

$\chi'(G) \geq \Delta(G)$ (since all edges incident to a vertex v must have distinct colors).

Theorem 16.3 — **König's Line Coloring Theorem**

In a bipartite graph G,

$$\chi'(G) = \Delta(G)$$

Theorem 16.4 — **Vizing**

For all graphs G,

$$\chi'(G) \leq \Delta(G) + 1$$

> **Note**
>
> This classifies all graphs into two classes:
> - **Class 1:** $\chi'(G) = \Delta(G)$ (e.g., Bipartite graphs).
> - **Class 2:** $\chi'(G) = \Delta(G) + 1$ (e.g., Odd cycles, K_n for odd n).

Lecture 26 (2026/01/27)

Proof of Theorem 16.3: Let $k = \Delta(G)$.

Step 1: Reduction to a k-regular bipartite graph.
If G is not k-regular, we can construct a k-regular bipartite graph H such that $G \subseteq H$ (analogously to Lemma 1.9):
First, add vertices to the smaller partition of G so that both partitions have equal size.
If the graph is still not k-regular, connect two vertices from different partitions that have degrees less than k. Repeat this until all vertices have degree k. (Note: This is always possible in a bipartite graph with equal partition sizes until regularity is achieved).

Any proper edge coloring of H restricted to G is a proper edge coloring of G. Thus, it suffices to prove the theorem for k-regular graphs.

Step 2: Coloring the k-regular graph.
Let H be a k-regular bipartite graph. By Lemma 6.13, H contains a perfect matching M_1. We assign color 1 to all edges in M_1.

Removing M_1 from H leaves a $(k-1)$-regular bipartite graph. We can repeat this process k times, finding perfect matchings $M_1, \ldots, M_k$. Assigning distinct colors to each matching yields a proper k-edge-coloring.

Therefore, $\chi'(G) \leq \chi'(H) \leq k = \Delta(G)$. Since $\chi'(G) \geq \Delta(G)$ always holds, we have $\chi'(G) = \Delta(G)$. □

Alternative proof of Theorem 16.3: Assume for a contradiction that $\exists$ bipartite graph G with $\chi'(G) \neq \Delta(G)$.
Let G be a minimal such graph (in terms of number of edges).
Let $uv = e \in E(G)$.

Consider $G - e$.
By minimality of G, $\chi'(G - e) = \Delta(G - e) \leq \Delta(G) =: k$.
Let $c : E(G - e) \to \{1, \ldots, k\}$ be a proper edge coloring of $G - e$.

Let $A = \{\text{colors used on edges incident to } u\}$ and $B = \{\text{colors used on edges incident to } v\}$.
Since $d_{G-e}(u), d_{G-e}(v) \leq k - 1$, $|A|, |B| \leq k - 1$.
Thus, $\exists$ color $\alpha \in \{1, \ldots, k\} \setminus A$ and color $\beta \in \{1, \ldots, k\} \setminus B$.

If $\alpha = \beta$, we can color e with α and obtain a proper k-edge-coloring of G, contradicting our assumption.

Suppose $\alpha \neq \beta$.
Consider the subgraph H of $G - e$ induced by edges colored α and β.
Since G is bipartite, all vertices in H have degree at most 2.
Therefore, H is a disjoint union of paths and even cycles.

Let P be the connected component of H containing u.
Since u is incident to an edge of color α but not β, P is a path that ends at u.
Let w be the other endpoint of P.

- If $w \neq v$:
  We can swap colors α and β along P.
  This frees up color α at u and color β at w.
  Now, we can color edge e with color α, yielding a proper k-edge-coloring of G, contradicting our assumption.
- If $w = v$:
  As P connects u and v, it must have odd length (since u and v are in different partitions of the bipartite graph).
  As c is a proper edge coloring, P must alternate between the colors α and β.
  Therefore, the starting edge and ending edge must have the same color, which is a contradiction as u and v don't share the colors α and β.

In both cases, we reach a contradiction. Therefore, our initial assumption is false, and $\chi'(G) = \Delta(G)$ for all bipartite graphs G. □

Definition 16.5 **Kempe chain**

Let G be a graph with a proper edge coloring. Let $v \in V(G)$ and colors a, b.
The ***ab*-Kempe chain** starting at v is the maximal connected subgraph of G containing v and only edges colored a or b.

Proof of Theorem 16.4: Assume G is an edge-minimal counterexample.
Let $k = \Delta(G)$.
Pick any edge $e = uv_0 \in E(G)$.

Consider $G - e$. By minimality of G, $\chi'(G - e) \leq k + 1$.
Let $c : E(G - e) \to \{1, \ldots, k + 1\}$ be a proper edge coloring of $G - e$. For any vertex x, let $M(x)$ be the set of colors in $\{1, \ldots, k + 1\}$ not appearing on edges incident to x. Since $d(x) \leq k$, $M(x) \neq \emptyset$.

Let $\alpha \in M(u)$. We construct a maximal sequence of distinct neighbors $v_0, v_1, \ldots, v_m$ of u (a "Fan") such that $c\left(uv_{\{i+1\}}\right) \in M(v_i)$ for all $0 \leq i < m$.

Let $\beta \in M(v_m)$. By the maximality of the fan, β is not missing at any v_i for $i < m$ (else we stop there), so β must appear on some edge incident to u. Thus, $\exists j \in \{1, \ldots, m - 1\}$ such that $c\left(uv_j\right) = \beta$.

Consider the the subgraph $H_{\alpha\beta}$ induced by edges colored α or β.

- **Case 1:** u and v_m are in different components of $H_{\alpha\beta}$.
  Switch colors α and β in the component containing v_m. Now $\alpha \in M(v_m)$ (and β is no longer missing at v_m). We can now switch the colors along the fan: set $c(uv_i) = c(uv_{i+1})$ for $i = 0, \ldots, m - 1$. Now the edge uv_m is uncolored, but since $\alpha \in M(v_m)$ (after the switch) and $\alpha \in M(u)$, we can color uv_m with α.

- **Case 2:** u and v_m are in the same component of $H_{\alpha\beta}$.
  There is an alternating path P between u and v_m. Since $\alpha \in M(u)$, the path must start with an edge colored β at u. The only such edge is uv_j. Thus P goes through v_j.

  We shift the fan only up to v_j: set $c(uv_i) = c(uv_{i+1})$ for $i = 0, \ldots, j - 1$. Now the uncolored edge is uv_j. Note that $\beta \in M\left(v_j\right)$ (by construction of the fan) and $\alpha \in M(u)$.

Crucially, removing the color from uv_j breaks the path P (since uv_j was the only β-edge at u). Now u and v_j are in different components of $H_{\alpha\beta}$. We can swap colors α and β in the component containing u. After swapping, $\beta \in M(u)$. Since $\beta \in M(v_j)$ still holds, we can color uv_j with β.

In both cases, we obtain a valid $(k+1)$-coloring of G, contradicting that G is a counterexample. □

17 Lower bounds for the independence number $\alpha(G)$

We investigate lower bounds for $\alpha(G)$ based on degree parameters.

> **Lemma 17.1**
>
> Let G be a graph with n vertices. Then,
>
> $$\alpha(G) \geq \frac{n}{\Delta(G)+1}$$

Proof: We construct an independent set I iteratively. Let $G_0 = G$ and $i = 0$. While $V(G_i) \neq \emptyset$:

1. Pick an arbitrary vertex $v \in V(G_i)$.
2. Add v to I.
3. Remove v and its neighbors: $G_{i+1} := G_i - \left(\{v\} \cup N_{G_i}(v)\right)$.
4. $i \leftarrow i + 1$.

In each step, we remove at most $d_G(v) + 1 \leq \Delta(G) + 1$ vertices. Thus, the process repeats at least $\frac{n}{\Delta(G)+1}$ times. Since we select one vertex per step, $|I| \geq \frac{n}{\Delta(G)+1}$. □

> **Lemma 17.2**
>
> Let G be a graph. Then,
>
> $$\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{d_G(v)+1}$$

Proof: Let $v_1, ..., v_n$ be the vertices of G.

Let $\sigma \in S_n$ (S_n being the symmetric group of permutations of $\{1, ..., n\}$).

Build an independent set S in the following way:

- Let $S = \left\{v_i \mid \forall j \in [\![n]\!] \text{ with } \sigma(j) < \sigma(i) : v_i v_j \notin E(G)\right\}$

Same as the following greedy algorithm:

- Mark $v_{\sigma^{-1}(1)}$ as being in S, delete it and its neighbors.
- Out of all vertices that remain, mark the vertex v_k with the smallest $\sigma(k)$, add it to S, and remove it and its neighbors.
- Repeat until no vertices are left.

Now we pick σ at random (each of the $n!$ possible choices have equal probability).

Define for each v the random variable $X_v := \begin{cases} 1 & \text{if } v \in S \\ 0 & \text{if } v \notin S \end{cases}$.

What is $\mathbb{P}(v \in S)$?

Consider the set consisting of v and its neighbors, $N(v) \cup \{v\}$. If v appears earlier in the permutation than all of its neighbors (i.e., $\sigma(\text{index}(v))$ is minimal in the set of ranks of $N(v) \cup \{v\}$), then v is definitely included in S because no neighbor could have eliminated it.

Since σ is random, any vertex in $N(v) \cup \{v\}$ is equally likely to be the first among them. Thus:

$$\mathbb{P}(v \text{ is first in } N(v) \cup \{v\}) = \frac{1}{d_G(v) + 1}$$

Since v might still be included even if it isn't first (e.g., if its earlier neighbors were eliminated by **their** predecessors), we have:

$$\mathbb{P}(v \in S) \geq \frac{1}{d_G(v) + 1}$$

$$\mathbb{E}(|S|) = \sum_{v \in V(G)} \mathbb{E}(X_v) = \sum_{v \in V(G)} \mathbb{P}(v \in S) \geq \sum_{v \in V(G)} \frac{1}{d_G(v) + 1}$$

□

Lecture 27 (2026/02/02)

Observation

We know that $\chi(G) \cdot \alpha(G) \geq v(G)$ (see Lemma 13.8).
Also, we know $\chi(G) \leq \Delta(G) + 1$ (see @Lemma 13.3)

Combining these yields the trivial lower bound:

$$\alpha(G) \geq \frac{1}{\Delta(G) + 1}$$

Q/ Is this the best possible?

Theorem 17.3 **Shearer '83**

Let G be a Δ-free ($K_3 \nsubseteq G$) graph with $n := v(G)$ vertices. Then,

$$\alpha(G) \geq \Omega\left(\frac{n \log \Delta(G)}{\Delta(G)}\right)$$

Note

This bound is best possible. The random $\Delta(G)$-regular graph has $\alpha(G) \sim \frac{2n \log \Delta(G)}{\Delta(G)}$ (for large $\Delta(G)$).

***Proof*:** Let $\Delta := \Delta(G)$
Let $\mathcal{J}$ be the set of all independent sets of G (including $\emptyset$).
Let W be a uniformly chosen independent set from $\mathcal{J}$.
For $v \in V(G)$, let $Y_v := \Delta \cdot |\{v\} \cap W| + |N_{G(v)} \cap W|$.

Observation:

$$\begin{aligned}
\sum_{v \in V(G)} Y_v &= \sum_{v \in V(G)} \Delta \mathbb{1}_{v \in W} + \sum_{v \in V(G)} \sum_{u \in N(v)} \mathbb{1}_{u \in W} \\
&= \Delta\ |W| + \sum_{u \in W} d_{G(u)} \\
&= \Delta\ |W| + e(G[W, W^c])
\end{aligned}$$

(Since W is independent, all neighbors of $u \in W$ must be in W^c).
Since $d_{G(u)} \leq \Delta$, we have $\sum_{u \in W} d_{G(u)} \leq \Delta \; |W|$. Thus:

$$\sum_{v \in V(G)} Y_v \leq \Delta \; |W| + \Delta \; |W| = 2\Delta \; |W|$$

Goal: Show $\mathbb{E}(Y_v) = \Omega(\log \Delta) \; \forall v \in V(G)$.
This then implies $2\Delta \mathbb{E}(|W|) \geq \sum_{v \in V(G)} \mathbb{E}(Y_v) = n\Omega(\log \Delta) \Rightarrow \mathbb{E}(|W|) \geq \Omega\left(\frac{n \log \Delta}{\Delta}\right)$.
This means there is a $W \in \mathcal{J}$ such that $|W| = \Omega\left(\frac{n \log \Delta}{\Delta}\right)$

Let $v \in V$.
Set $N[v] := N(v) \cup \{v\}$ (the closed neighborhood of v).
Let $H := G - N[v]$

For a fixed independent set $S \subseteq V(H)$, define the set of available neighbors:

$$X := X(S) := N(v) \setminus \bigcup_{u \in S} N(u)$$

Let $x := |X|$.

Lemma 17.4

Let J be an independent set such that $J \subseteq X \cup \{v\}$. Then,

$$\mathbb{P}(W \cap N[v] = J \mid W \cap V(H) = S) = \frac{1}{2^x + 1}$$

Proof: The total number of independent sets I such that $I \cap V(H) = S$ is determined by how S can be extended into $N[v]$.

1. We can pick v. Since $S \subseteq V(H)$, S is not connected to v. This gives $J = \{v\}$.
2. We can pick $v \notin W$. Then we can pick any subset of neighbors $J \subseteq N(v)$. For $J \cup S$ to be independent, J must not connect to S. Also $N(v)$ is independent (triangle-free). Thus, any $J \subseteq X$ works. There are 2^x such subsets.

Total extensions = $2^x + 1$. Each is equally likely under uniform distribution. □

Corollary 17.5

$$\mathbb{E}\left(|\{v\} \cap W| \; \mathbb{1}_{\{V(H) \cap W\}}\right) = \frac{1}{2^x + 1} \mathbb{P}(W \cap V(H) = S)$$

Proof:

$$\begin{aligned}
\mathbb{E}\left(|\{v\} \cap W| \; \mathbb{1}_{\{V(H) \cap W\}}\right) &= \mathbb{P}(v \in W, V(H) \cap W = S) \\
&= \mathbb{P}(W = \{v\} \cup S) \\
&= \mathbb{P}(W \cap N[v] = \{v\} \mid W \cap V(H) = S) \cdot \mathbb{P}(W \cap V(H) = S) \\
&= \frac{1}{2^x + 1} \mathbb{P}(W \cap V(H) = S)
\end{aligned}$$

□

Corollary 17.6

$$\mathbb{E}\left(|W \cap N(v)|\, \mathbb{1}_{\{V(H)\cap W = S\}}\right) = \frac{x2^{x-1}}{2^x+1}\mathbb{P}(W \cap V(H) = S)$$

Proof:

$$\begin{aligned}\mathbb{E}(|W \cap N(v)| \mid S) &= \sum_{j=0}^{x} \sum_{J\subseteq X,\ |J|=j} j \cdot \mathbb{P}(W \cap N[v] = J \mid S)\\ &= \sum_{j=0}^{x}\binom{x}{j}\cdot j \cdot \frac{1}{2^x+1}\\ &= \frac{1}{2^x+1}\sum_{j=1}^{x} x\binom{x-1}{j-1}\\ &= \frac{x2^{x-1}}{2^x+1}\end{aligned}$$

Multiplying by $\mathbb{P}(W \cap V(H) = S)$ gives the result. □

Combining the corollaries:

$$\mathbb{E}(Y_v \mid S) = \frac{\Delta + x2^{x-1}}{2^x+1}$$

Claim 17.7

For $x \geq 0$ and $\Delta \geq 1$:

$$\frac{\Delta + x2^{x-1}}{2^x+1} \geq \frac{\log_2 \Delta}{4}$$

Proof: Let $f(x) := \frac{\Delta+x2^{x-1}}{2^x+1}$. Note that $2^x + 1 \leq 2^{x+1}$.

$$f(x) \geq \frac{\Delta}{2^{x+1}} + \frac{x}{4}$$

We distinguish two cases:

1. If $x < \frac{\log_2 \Delta}{2}$:
   Then $x + 1 < \frac{\log_2 \Delta}{2} + 1$. For large Δ, $2^{x+1} \leq 2 \cdot \sqrt{\Delta}$.

$$f(x) \geq \frac{\Delta}{2\sqrt{\Delta}} = \frac{\sqrt{\Delta}}{2} \geq \frac{\log_2 \Delta}{4}$$

.

2. If $x \geq \frac{\log_2 \Delta}{2}$:
   Then simply $f(x) \geq \frac{x}{4} \geq \frac{\log_2 \Delta}{8}$.

In both cases, $f(x) = \Omega(\log \Delta)$. □

Finally, taking the expectation over S:

$$\mathbb{E}(Y_v) = \sum_S \mathbb{E}(Y_v \mid S)\mathbb{P}(W \cap V(H) = S) \geq \min_S \mathbb{E}(Y_v \mid S) = \Omega(\log \Delta)$$

This completes the proof. □

18 Extremal graph theory

Q/ What is the maximum number of edges in a Δ-free graph on $n := v(G)$ vertices?

Theorem 18.1 **Mantel**

Let G be a Δ-free graph with $n := v(G)$ vertices. Then,

$$e(G) \leq e\left(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}\right) = \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil$$

Moreover, equality holds if and only iff $G \cong K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

Lecture 28 (2026/02/03)

Proof: Let $v \in V(G)$ with $d_{G(v)} = \Delta(G)$. Let $B := N(v)$ and $A := V(G) \setminus B$.

Note: As G is Δ-free (triangle-free), B must be an independent set (otherwise two neighbors of v would be connected, forming a triangle with v). $\Rightarrow e(G[B]) = 0$. Thus, all edges lie within A or between A and B: $e(G) = e(G[A]) + e(G[A, B])$.

Summing the degrees of vertices in A:

$$\sum_{u \in A} d_{G(u)} = 2e(G[A]) + e(G[A, B]) = e(G[A]) + e(G)$$

We also know $d_{G(u)} \leq \Delta(G)$ for all u, and $|A| = n - |B| = n - \Delta(G)$.

$$\sum_{u \in A} d_{G(u)} \leq |A| \ \Delta(G) = (n - \Delta(G))\Delta(G)$$

Combining these inequalities:

$$e(G) + e(G[A]) \leq \Delta(G)(n - \Delta(G))$$

Since $e(G[A]) \geq 0$, we have:

$$e(G) \leq \Delta(G)(n - \Delta(G))$$

Consider the function $f(x) = x(n - x)$. This is a downward-facing parabola maximized at $x = \frac{n}{2}$. Therefore:

$$e(G) \leq \left(\frac{n}{2}\right)\left(n - \frac{n}{2}\right) = \frac{n^2}{4}$$

Since $e(G)$ must be an integer, $e(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor = \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$.

#

Equality: For equality to hold ($e(G) = \left\lfloor \frac{n^2}{4} \right\rfloor$), all inequalities used above must be equalities:

1. $e(G[A]) = 0$ (so A is also an independent set).
2. $d_{G(u)} = \Delta(G)$ for all $u in A$.
3. $\Delta(G)(n - \Delta(G)) = \left\lfloor \frac{n^2}{4} \right\rfloor$, which implies $\Delta(G) \in \{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil\}$.

From (1), since both A and B are independent sets, G is bipartite. From (2) and (3), every vertex in A is connected to every vertex in B (since $|B| \approx \frac{n}{2}$). Thus, G must be the complete bipartite graph $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. □

Definition 18.2

Let H be a graph and $n \in \mathbb{N}$. Then, we call

$$\operatorname{ex}(H, n) := \max\{e(G) \mid \text{graph } G, v(G) = n, H \nsubseteq G\}$$

the extremal number of H.

Q/ What is a lower bound to $\operatorname{ex}(K_r, n)$?

Definition 18.3 **Turán graph**

For $r, n \in \mathbb{N}_{\geq 2}$, the Turán graph $T_{r,n}$ is the n-vertex graph created as follows:
- Let $V(T_{r,n})$ be partitioned into r parts $V_1, \ldots, V_r$ with as equal size as possible (i.e., $|V_i| - |V_j| \leq 1$ for all i, j).
- $E(T_{r,n}) = \bigcup_{j \neq i} \{uv \mid u \in V_i, v \in V_j\}$

Theorem 18.4 **Turán's Theorem**

For all $r \geq 3$, $\operatorname{ex}(K_r, n) = e(T_{r-1,n})$

Moreover, if G is a K_r-free graph on n vertices with $e(G) = e(T_{r-1,n})$, then $G \cong T_{r-1,n}$

Proof: Let G be a graph on n vertices which is K_r-free.
Take $v \in V(G)$ with $d_G(v) = \Delta(G)$.
Let $B := N(v)$ (so $|B| = \Delta(G)$).
Let $A := V(G) \setminus B$ (so $|A| = n - \Delta(G)$).

Since G is K_r-free, the neighborhood B cannot contain a K_{r-1} (otherwise with v it forms K_r). By induction, $e(G[B]) \leq e(T_{r-2,\Delta(G)})$.

$$\begin{aligned} e(G) &= e(G[B]) + e(G[A, B]) + e(G[A]) \\ &= e(G[B]) + \sum_{u \in A} d_G(u) - e(G[A]) \\ &\leq e(T_{r-2,\Delta(G)}) + |A| \, \Delta(G) - e(G[A]) \\ &\leq e(T_{r-2,\Delta(G)}) + (n - \Delta(G))\Delta(G) \\ &= e(H) \end{aligned}$$

where H is the graph constructed by taking $T_{r-2,\Delta(G)}$ on B, an independent set on A, and connecting every vertex in A to every vertex in B. Note that H is a complete $(r-1)$-partite graph (partitions of B plus the set A).

Lemma 18.5

Let H be a complete $(r-1)$-partite graph on n vertices. Then $e(H) \leq e(T_{r-1,n})$. Moreover, equality holds $\Leftrightarrow H \cong T_{r-1,n}$.

Proof: Maximizing edges in a complete multipartite graph is equivalent to minimizing edges in its complement H^c (which is a union of disjoint cliques).

Goal: $e(H^c) \geq e(T^c_{r-1,n})$.

Let $V_1, \ldots, V_{r-1}$ denote the vertex sets of the disjoint cliques in H^c. Suppose the partition is not balanced, i.e., there exist V_1, V_2 such that $|V_1| \geq |V_2| + 2$.

Move a vertex x from V_1 to V_2. The change in the number of edges in H^c is:

$$|V_2| - (|V_1| - 1) = |V_2| - |V_1| + 1 \leq |V_2| - (|V_2| + 2) + 1 = -1$$

This operation strictly decreases the number of edges in H^c (and thus strictly increases edges in H). Repeating this process until all parts differ by at most 1 yields $T_{r-1,n}$. Thus, $e(H) \leq e(T_{r-1,n})$ with equality if and only if the partitions were already balanced ($H \cong T_{r-1,n}$). □

Combining the main inequality with the lemma, we have $e(G) \leq e(H) \leq e(T_{r-1,n})$.

Equality: If $e(G) = e(T_{r-1,n})$, all inequalities must be equalities.
1. $e(G[A]) = 0$ (A is independent).
2. $d_{G(u)} = \Delta(G)$ for all $u \in A$. Since all edges from A must go to B, A is fully connected to B.
3. $e(G[B]) = e\left(T_{r-2,\ |B|}\right)$. By induction, $G[B] \cong T_{r-2,\ |B|}$.

Thus, G is formed by a Turán graph on B fully connected to an independent set A. This makes G a complete $(r-1)$-partite graph. By the Lemma, among such graphs, only $T_{r-1,n}$ achieves the maximum number of edges. $\Rightarrow G \cong T_{r-1,n}$. □

Theorem 18.6 **Erdös-Stone**

For any graph H, $\operatorname{ex}(H,n) = e\left(T_{\chi(H)-1,n}\right) \cdot (1 + o(1))$. Explicitly,

$$\operatorname{ex}(H,n) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}$$

Proof: in Discrete Structures 2 □

Q/ What is the right behavior of $\operatorname{ex}(n, H)$ when H is bipartite?

Note

$\operatorname{ex}(K_{1,1}, n) = 0$

$\operatorname{ex}(K_{1,2}, n) = \Theta(n)$

$\operatorname{ex}(K_{1,3}, n) = \Theta(n)$

$\operatorname{ex}(P_4, n) = \Theta(n)$

Theorem 18.7

$$\operatorname{ex}(C_4, n) = O\left(n^{\frac{3}{2}}\right)$$

Proof: Let G be a C_4-free graph with n vertices and m edges. We count the number of "cherries" (paths of length 2, i.e., triples (u, v, w) such that $uv, vw \in E(G)$) in two ways.

1. **Summing over the middle vertex:** Every vertex v is the center of exactly $\binom{d(v)}{2}$ cherries.

$$\#\text{ cherries} = \sum_{v \in V(G)} \binom{d_G(v)}{2}$$

2. **Summing over the endpoints:** A cherry corresponds to a pair of vertices $\{u, w\}$ connected by a common neighbor v. Since G is C_4-free, any pair of vertices $\{u, w\}$ has at most **one** common neighbor (otherwise, two common neighbors would form a C_4).

$$\# \text{ cherries} \leq \binom{n}{2}$$

Combining these gives:

$$\sum_{v \in V(G)} \binom{d_G(v)}{2} \leq \binom{n}{2}$$

We apply **Jensen's Inequality**. The function $f(x) = \binom{x}{2} = \frac{x(x-1)}{2}$ is convex. Let $\overline{d} = \frac{2m}{n}$ be the average degree.

$$\sum_{v \in V(G)} f(d(v)) \geq n \cdot f\left(\frac{\sum d(v)}{n}\right) = n \cdot \binom{\overline{d}}{2}$$

Substituting this back:

$$\begin{aligned} n \cdot \frac{\frac{2m}{n}\left(\frac{2m}{n} - 1\right)}{2} &\leq \frac{n(n-1)}{2} \\ \frac{2m}{n}\left(\frac{2m}{n} - 1\right) &\leq n - 1 \\ \frac{2m}{n} \cdot \frac{2m}{n} &\leq n \qquad \text{(for large n, ignoring -1 term)} \\ \frac{4m^2}{n^2} &\leq n \\ m^2 &\leq \frac{n^3}{4} \end{aligned}$$

Thus, $e(G) = m \leq \frac{1}{2} n^{\frac{3}{2}} = O\left(n^{\frac{3}{2}}\right)$. □

Note **Optimality of the bound**

The bound $O\left(n^{\frac{3}{2}}\right)$ is tight. We can construct C_4-free graphs with $\Omega\left(n^{\frac{3}{2}}\right)$ edges using finite geometry.

Intuition ($\mathbb{R}^2$): Consider a bipartite graph where $A =$ Points in $\mathbb{R}^2$ and $B =$ Lines in $\mathbb{R}^2$. Connect a point p to a line l if $p \in l$.

- Two points determine a unique line $\Rightarrow$ No two points share $ge2$ common line neighbors.
- Two lines intersect in a unique point $\Rightarrow$ No two lines share $ge2$ common point neighbors.

This implies the graph is C_4-free. However, this graph is infinite.

Formal Construction (Finite Projective Plane): Let $PG(2, q)$ be the projective plane over the finite field $\mathbb{F}_q$.

- Vertices: Let G be the incidence graph with vertices $V = \mathcal{P} \cup \mathcal{L}$ (Points and Lines).
- Size: $|\mathcal{P}| = |\mathcal{L}| = q^2 + q + 1$. Thus $n = 2\left(q^2 + q + 1\right) \approx 2q^2$.
- Edges: Each line contains $q + 1$ points; each point lies on $q + 1$ lines. The graph is $(q + 1)$-regular. $e(G) = |\mathcal{P}| \ (q + 1) \approx q^3$.

Check C_4-freeness: In a projective plane, any two distinct lines intersect in exactly one point, and any two distinct points lie on exactly one line. Thus, no $K_{2,2}$ (C_4) exists.

Result: Since $n \approx 2q^2$, we have $q \approx \sqrt{\frac{n}{2}}$. The number of edges is $e(G) \approx q^3 \approx \left(\frac{n}{2}\right)^{\frac{3}{2}} = \Omega\left(n^{\frac{3}{2}}\right)$.